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BY

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TERMS.

34. The **Terms** of an algebraic quantity are the divisions made by the signs, + and —; thus, in the quantity $3a + b^2 - mx$, there are three terms, of which $3a$ is the first, $+ b^2$ is the second, and $- mx$ is the third.

35. **Positive Terms** are those which have the *plus* sign; as, $+ a$ or $+ b^2d$. The first term of an algebraic quantity, if written without any sign, is positive, the sign + being understood.

36. **Negative Terms** are those which have the *minus* sign; as, $- 2a$, or $- 3c^2d$. The sign of a negative quantity is never omitted.

37. **Similar Terms** are terms containing the same letters, affected with the same exponents; the signs and coefficients may differ, and the terms still be similar. Thus, $3a^2$ and $5a^2$ are similar terms; $2b^2d$ and $- 5b^2d$ are similar terms.

38. **Dissimilar Terms** are those which have different letters or exponents; thus, abc and acd are dissimilar terms; ax^2y^3 and a^2xy are dissimilar terms.

39. A **Monomial** is an algebraic quantity consisting of only one term; as, $4a$, $3cd$, or $7b^2x$.

40. A **Polynomial** is an algebraic quantity consisting of more than one term; $a + b$; or $3ab - 2x + c$.

41. A **Binomial** is a polynomial of two terms; as $a + c$, or $2x - y$.

42. A **Residual** is a binomial, the two terms of which are connected by the *minus* sign; as, $a - b$, or $3x - 2y$.

43. A **Trinomial** is a polynomial of three terms; as $x + y + z$, or $3a - 2b + c^2$.

44. The **Degree** of a term is the number of literal factors it contains, and is found by adding the exponents of the several letters; thus, a and $3b$ are terms of the first degree; a^2 and

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Define the terms of an algebraic quantity. Positive terms. Negative terms. Similar terms. Dissimilar terms. A Monomial. A Polynomial. A Binomial. A Residual. The Degree of a term.

$2ab$ are terms of the second degree; a^3, $3a^2b$, and $5abc$ are terms of the third degree.

45. A **Homogeneous quantity** is one whose terms are all of the same degree; as, $x^3 - 3x^2y + xyz$.

AXIOMS.

46. An **Axiom** is a self-evident truth.

The principles of all algebraic operations are based upon the following axioms:

1. If the same quantity or equal quantities be *added* to equal quantities, their *sums* will be equal.
2. If the same quantity or equal quantities be *subtracted* from equal quantities, the *remainders* will be equal.
3. If equal quantities be multiplied by the same, or equal quantities, the *products* will be equal.
4. If equal quantities be *divided* by the same, or by equal quantities, the *quotients* will be equal.
5. If the same quantity be both *added to* and *subtracted from* another, the value of the latter will not be altered.
6. If a quantity be both *multiplied* and *divided* by another, the value of the former will not be altered.
7. Quantities which are respectively equal to any other quantity are equal to each other.
8. Like powers of equal quantities are equal.
9. Like roots of equal quantities are equal.
10. The whole of any quantity is greater than any of its parts.
11. The whole of any quantity is equal to the sum of all its parts.

Define a Homogeneous Quantity. An Axiom. Repeat the Axioms given.

ADDITION.

47. **Addition,** in Algebra, is the process of uniting two or more quantities into one equivalent expression, called their sum.

Since, in algebra, the quantities to be added may be either positive or negative, it is necessary to consider here more fully the nature of the signs + and —. Thus far they have been employed to indicate simply the opposite processes of addition and subtraction. They have, however, a wider significance, and indicate not only *operations to be performed,* but the *quality,* or *relative character* of the quantities to which they are applied. They may denote opposite directions in space, opposite effects in nature, or opposite results in business. Thus, if *plus* indicate direction north, *minus* will indicate direction south; if *plus* indicate heat, *minus* will indicate cold; and if *plus* indicate gain, *minus* will indicate loss.

CASE I.

48. To add similar terms.

1. A cooper made 7 barrels on Monday, 9 barrels on Tuesday, and 6 barrels on Wednesday; how many barrels did he make in the three days?

ARITHMETICALLY.

7 barrels.
9 barrels.
6 barrels.

Ans. 22 barrels.

Define Addition. Explain the nature of the signs + and —. What is Case I? Give Analysis.

ALGEBRAICALLY.

$$\begin{array}{r} 7b \\ 9b \\ 6b \\ \hline 22b \end{array}$$

ANALYSIS. We represent 1 barrel by the letter b; then $7b$ will represent 7 barrels, $9b$ will represent 9 barrels, and $6b$ will represent 6 barrels; and since 7 barrels, 9 barrels, and 6 barrels are 22 barrels, $7b$, $9b$, and $6b$ are $22b$.

2. A mass of iron and wood is submerged in water by its own gravity. The iron tends to sink with a force of $20l$, and the wood buoys upward with a force of $16l$; what will the whole mass weigh while under water?

OPERATION.

$$\begin{array}{r} +20l \\ -16l \\ \hline +\ \ 4l \end{array}$$

ANALYSIS. We indicate actual weight by the *plus* sign, and the opposite force, or buoyancy, by the *minus* sign. Since the tendency to sink is greater by $4l$ than the tendency to rise, the mass has a weight of $4l$; hence, $+20l$ and $-16l$ united are $+4l$.

NOTE.—The answer, $+4l$, in the above example, is called the *algebraic sum* of the two forces, $+20l$ and $-16l$, because it shows their *united effect.*

3. A ship started at the equator and sailed the first day 16 miles north, the second day 20 miles south, the third day 8 miles north, and the fourth day 7 miles south; how far from the equator, and in what latitude, was the ship at the end of the four days?

FIRST OPERATION.

$$\begin{array}{rr} +16m & -20m \\ +\ \ 8m & -\ \ 7m \\ \hline +24m & -27m = -3m \end{array}$$

ANALYSIS. We let m represent 1 mile; and to distinguish the directions, we indicate distance north by the *plus* sign, and distance south by the *minus* sign, writing the positive terms in one column and the negative terms in another column. The whole distance sailed north, is $+16m$ and $+8m$, which is $+24m$; and the whole distance sailed south, is $-20m$ and $-7m$, which is $-27m$; and, as 27 is 3 more than 24, the ship must be three miles south of the equator, expressed algebraically thus, $-3m$.

Explain the difference between Arithmetical and Algebraic Addition, as shown in example 3.

SECOND OPERATION.
$+16m$
$-20m$
$+\ 8m$
$-\ 7m$
$-\ 3m$

ANALYSIS. In the second operation we write all the terms in one column, since they are similar. $+16m$ and $+8m$ are $+24m$; and $-20m$ and $-7m$ are $-27m$; and $+24m - 27m$ are $-3m$, the algebraic sum of the quantities in the column.

NOTE.—There is a distinction between *arithmetical* and *algebraic* addition. In the above example had the question been, *how far the ship sailed*, the answer would be $16 + 20 + 8 + 7 = 51$ miles, or $51m$, which is the *arithmetical* sum of the distances sailed. But the real question is, *what distance, north or south, did the ship make from the point of starting;* and the answer is 3 miles south, or $-3m$, which is the *algebraic* sum of the distances sailed. Hence, adding, in algebra, does not always augment. Positive and negative quantities represent things opposite in kind or quality, and, if similar in *denomination*, are added or united, by *apparent subtraction.*

From these examples and illustrations we derive the following

RULE. I. *When the signs are alike, add the coefficients, and prefix the sum with its proper sign to the common literal part.*

II. *When the signs are unlike, find the sum of the positive and of the negative coefficients separately, and prefix the difference of the two sums with the sign of the greater, to the common literal part.*

EXAMPLES FOR PRACTICE.

(4.)	(5.)	(6.)	(7.)	(8.)
$3a$	$2m^2$	$-\ 3bx$	$-\ 4a^2bc$	$+\ 7cd^2$
$9a$	$6m^2$	$-\ 5bx$	$-\ 5a^2bc$	$+\ 3cd^2$
$5a$	$5m^2$	$-\ 4bx$	$-12a^2bc$	$+\ 2cd^2$
$12a$	$10m^2$	$-\ 2bx$	$-\ a^2bc$	$+\ cd^2$
a	$5m^2$	$-\ 7bx$	$-14a^2bc$	$+\ 6cd^2$
$2a$	$7m^2$	$-\ bx$	$-\ 2a^2bc$	$+\ 4cd^2$
$32a$		$-22bx$		$+23cd^2$

Give the rule for the algebraic addition of similar terms.

(9.)	(10.)	(11.)	(12.)	(13.)
$-5a$	$+3ax^2$	$+8x^3$	$-5a^2$	$+3b^2y^3$
$+4a$	$+4ax^2$	$-5x^3$	$-10a^2$	$+9b^2y^3$
$+6a$	$-8ax^2$	$-16x^3$	$+10a^2$	$-10b^2y^3$
$-3a$	$-6ax^2$	$+3x^3$	$+14a^2$	$-19b^2y^3$
$+a$	$+5ax^2$	$+2x^3$	$+6a^2$	$-2b^2y^3$
$+3a$	$-2ax^2$	$-8x^3$	$+15a^2$	$-19b^2y^3$

(14.)	(15.)	(16.)	(17.)
$5b^2d$	$2mn$	$-25a^2bc$	$147z^3$
$-7b^2d$	$4mn$	$36a^2bc$	$-25z^3$
$4b^2d$	$-12mn$	$-72a^2bc$	$12z^3$
$-3b^2d$	$16mn$	$48a^2bc$	$-14z^3$

18. What is the sum of $5m$, $7m$, $11m$, m, and $12m$?
Ans. $36m$.

19. What is the sum of $4a$, $7a$, $6a$, and $10a$?
Ans. $27a$.

20. What is the sum of $-5c^2$, $10c^2$, $-28c^2$, and $-c^2$?
Ans. $-24c^2$.

21. What is the sum of $-125bc$, $-168bc$, and $2bc$?
Ans. $-291bc$.

22. What is the sum of $3xy$, $-4xy$, $10xy$, and $-7xy$?
Ans. $2xy$.

23. What is the sum of $9a^2bc$, $3a^2bc$, $-8a^2bc$, $-2a^2bc$, $5a^2bc$, and $-15a^2bc$? *Ans.* $-8a^2bc$.

24. Add $5m^2x$, $-2m^2x$, $-7m^2x$, and $28m^2x$.
Ans. $24m^2x$.

25. Add $7x^2y^2$, $-15x^2y^2$, $+12x^2y^2$, $-6x^2y^2$, and x^2y^2.
Ans. $-x^2y^2$.

In the same manner similar quantities, of whatever kind, may be added by taking the algebraic sum of their coefficients. The pupil will observe that, since 2 times any num-

ber whatever, added to three times the same number, are 5 times that number, so $2(a+b)$, added to $3(a+b)$, are $5(a+b)$.

(26.)	(27.)	(28.)	(29.)
$4(c-x)$	$(x+y)$	$-4(2a-b)$	$4(x-y+3)$
$7(c-x)$	$-3(x+y)$	$-7(2a-b)$	$7(x-y+3)$
$10(c-x)$	$20(x+y)$	$8(2a-b)$	$-12(x-y+3)$
$21(c-x)$	$18(x+y)$	$-3(2a-b)$	$-(x-y+3)$

(30.)	(31.)	(32.)
$3a(a+b)$	$7(6x+y-z)^2$	$4(6y+b)$
$7a(a+b)$	$-8(6x+y-z)^2$	$-3(6y+b)$
$-5a(a+b)$	$-2(6x+y-z)^2$	$7(6y+b)$
$3a(a+b)$	$3(6x+y-z)^2$	$-2(6y+b)$

33. What is the sum of $3(z-m)$, $5(z-m)$, $-12(z-m)$, $-14(z-m)$, and $10(z-m)$? *Ans.* $-8(z-m)$.

34. What is the sum of $-4(a+2b)^2$, $5(a+2b)^2$, $-12(a+2b)^2$, and $20(a+2b)^2$? *Ans.* $9(a+2b)^2$.

35. What is the sum of $(x+1)$, $5(x+1)$, $3(x+1)$, and $-8(x+1)$? *Ans.* $(x+1)$.

CASE II.

49. To add polynomials.

It is evident that dissimilar terms may be added by writing them one after another, connected by their proper signs; but, if in the same polynomial, or in the different polynomials to be added, there be similar terms, these may be united by Case I, and a reduced expression obtained; and it is immaterial in what order the terms are written in the aggregate sum, since the whole is equal to the sum of all its parts, in whatever order the parts are taken, (Ax. 11).

1. Add $3a+2bc$, $4a-7bc+m$, and $x^2-2a+3bc$.

What is Case II?

OPERATION.

$$\begin{array}{l} 3a + 2bc \\ 4a - 7bc + m \\ x^2 - 2a + 3bc \\ \hline x^2 + 5a - 2bc + m \end{array}$$

ANALYSIS. We write $3a$, $4a$, and $-2a$, in one column, because they are similar terms, and $2bc$, $-7bc$, and $3bc$, in another column, for the same reason; and x^2 and m not being similar to each other, or to any of the other terms, we write them in separate columns. Commencing at the left, we write x^2 in the sum; then $-2a$, $+4a$, and $+3a$ are $+5a$, which we write under the column added; and $+3bc$, $-7bc$, and $+2bc$, are $-2bc$, which we write under the column added; finally annexing $+m$, we obtain the entire sum, $x^2 + 5a - 2bc + m$.

From this example we deduce the following

RULE. I. *Write similar terms in the same column, forming as many columns as there are dissimilar terms in the given quantities.*

II. *Add each column, as in case I, and connect the results by their proper signs*

EXAMPLES FOR PRACTICE.

$$\begin{array}{r} (2.) \\ 3x^2 - 4cd \\ 7x^2 - 8cd \\ -5x^2 + 9cd \\ \hline 5x^2 - 3cd \end{array} \qquad \begin{array}{r} (3.) \\ 4x^2y - a^2b^2 \\ 3x^2y + 9a^2b^2 - m \\ z - 5x^2y - 12a^2b^2 \\ \hline z + 2x^2y - 4a^2b^2 - m \end{array} \qquad \begin{array}{r} (4.) \\ 5ab^2 - 7dc \\ 7ab^2 + 14dc \\ -12ab^2 - 6dc \\ \hline + dc \end{array}$$

5. What is the sum of $6ab + 12bc - 8cd$, $3cd - 7ab - 9bc$, and $12cd - 2ab - 5bc$? *Ans.* $7cd - 3ab - 2bc$.

6. What is the sum of $9b^2 - 3ac + d$, $4b^2 + 7d - 4ac$, $3d - 4b^2 + 6ac$, $5b^2 - 2ac - 12d$, and $4b^2 - d$?

Ans. $18b^2 - 3ac - 2d$.

7. What is the sum of $7ab - m^2 + q$, $-4ab - 5m^2 - 8q$, $12ab + 14m^2 - z$, and $-6m^2 - 2q$?

Ans. $15ab + 2m^2 - 4q - z$.

Give analysis. Rule.

8. What is the sum of $6x - 5b + a + 8$, and $-5a - 4x + 4b - 3$? *Ans.* $2x - b - 4a + 5$.

9. What is the sum of $a + 2b - 3c - 10$, $3b - 4a + 5c + 10$, and $5b - c$? *Ans.* $-3a + 10b + c$.

10. What is the sum of $3a + b - 10$, $c - d - a$, and $-4c + 2a - 3b - 7$? *Ans.* $4a - 2b - 3c - d - 17$.

11. Add $15a^2 - 8b^2c + 32a^2c^3 - 12bc$, $19b^2c - 4a^2 + 11a^2c^3 + 2bc$, $a^2 - 29a^2c^3 - 12b^2c + 5bc$, and $9a^2c^3 - 14bc + b^2c$. *Ans.* $12a^2 + 23a^2c^3 - 19bc$.

12. Add $5a^3b^2 - 8a^2b^3 + x^2y + xy^2$, $4a^2b^3 - 7a^3b^2 - 3xy^2 + 6x^2y$, $3a^3b^2 + 3a^2b^3 - 3x^2y + 5xy^2$, and $2a^2b^3 - a^3b^2 - 3x^2y - 3xy^2$. *Ans.* $a^2b^3 + x^2y$.

13. Add $72ax^4 - 8ay^3$, $-38ax^4 - 3ay^4 + 7ay^3$, $8 + 12ay^4$, $-6ay^3 + 12$, and $-34ax^4 + 5ay^3 - 9ay^4$.
Ans. $-2ay^3 + 20$.

14. Add $7x^2 - 5cx + 14mg$, $-3x^2 + 4cx - 17mg - pq$, $4x^2 + 12mg + 3pq - z$, $2cx - 7mg - 2pq$, and $3x^2 - 2cx - mg - 4pq + 3z$. *Ans.* $11x^2 - cx + mg - 4pq + 2z$.

15. Add $7m + 3n - 11p$, $3a - 9n - 11m$, $8n - 4m + 5p$, and $6n - m + 3p$. *Ans.* $3a - 9m + 8n - 3p$.

16. Add $7a - 3b + c + m$, and $3b - 7a - c + m$.
Ans. $2m$.

17. Add $x - y - z$, and $y - x + z$. *Ans.* 0.

18. Add $3(a + b)$, $4(a + b)$, and $-2(a + b)$.
Ans. $5(a + b)$.

19. Add $6(m^2 - n) + 2c$, $-5(m^2 - n) + 7c$, $3(m^2 - n) - 4c$, and $4(m^2 - n) + c$. *Ans.* $8(m^2 - n) + 6c$.

20. Add $2a(x - y^2) - 3mz^2$, $4a(x - y^2) - 5mz^2$, and $5a(x - y^2) + 7mz^2$. *Ans.* $11a(x - y^2) - mz^2$.

21. Add $8ax + 2(x + a) + 3b$, $9ax + 6(x + a) - 9b$, and $11x + 6b - 7ax - 8(x + a)$. *Ans.* $10ax + 11x$.

50. The **Unit of addition** is the quantity whose coefficients are added; thus, in the example, $3x - 4x + 7x = 6x$, the unit of addition is x; for the result, $6x$, was obtained by uniting the coefficients of x into one number. Dissimilar terms may often be added, by making some common letter or letters the unit of addition.

1. What is the sum of ax, bx, and cx?

OPERATION.

$$\begin{array}{lr} & ax \\ & bx \\ & cx \\ \hline \text{Sum.} & (a+b+c)x \end{array}$$

ANALYSIS. We make the common letter, x, the unit of addition; thus a times x, b times x, and c times x must be equal to x multiplied by the *sum* of a, b, and c; and since a, b, and c are dissimilar, we *indicate* their addition, inclose the sum in a parenthesis, and write it as the coefficient of x, and thus obtain the sum of the given quantities.

EXAMPLES FOR PRACTICE.

(2.)	(3.)	(4.)
ax	by^2	$7ay$
$2cx$	$3ay^2$	$-2ay$
$4dx$	$7y^2$	$-cy$
$(a+2c+4d)x$	$(b+3a+7)y^2$	$(5a-c)y$

5. What is the sum of cx, $2cx$ and $6x$, when x is the unit of addition? *Ans.* $(3c+6)x$.

6. What is the sum of am^2, $-bm^2$, $+cm^2$, when m^2 is the unit of addition? *Ans.* $(a-b+c)m^2$.

7. What is the sum of $(a+b)x$ and $(a+c)x$, when x is the unit of addition? *Ans.* $(2a+b+c)x$.

8. What is the sum of $3x$, bx, and $(a+b)x$, when x is the unit of addition? *Ans.* $(a+2b+3)x$.

9. What is the sum of $4axy$, $-axy$, and $+cxy$, when xy is the unit of addition? *Ans.* $(3a+c)xy$.

Define the Unit of Addition.

SUBTRACTION.

51. Subtraction, in algebra, is the process of finding the difference between two quantities.

CASE I.

52. To find the difference of similar terms.

1. A and B travel north from the same point; the distance A travels is $7m$, and the distance B travels is $4m$; how much farther north is A than B?

OPERATION.

Minuend,	$7m$
Subtrahend,	$4m$
Difference,	$3m$

ANALYSIS. A must be as much farther north than B, as B's distance, $4m$, subtracted from A's distance, $7m$; and $7m - 4m = 3m$, the answer.

2 A and B start from the same point; A travels north a distance of $7m$, and B travels south a distance of $4m$; how much farther north is A than B?

OPERATION.

Minuend	$+ 7m$
Subtrahend,	$- 4m$
Difference,	$+ 11m$

ANALYSIS. To express the distances algebraically, we indicate the different directions by opposite signs; north by *plus*, south by *minus*. Since A traveled $7m$ north, while B traveled $4m$ south, A must be $11m$ farther north than B, and to indicate his direction from B we must use the plus sign, thus $+ 11m$.

The expression, $+ 11m$, in the last example, is called the *algebraic difference* of $+ 7m$ and $-4m$, because it denotes their *distance asunder*. To subtract, in algebra, is not in all cases to diminish. A positive and a negative quantity are in

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Define Subtraction. What is Case I?

opposite circumstances, or counted in opposite directions; *hence the difference, or space between them, is their apparent sum.* If we demand the difference of latitude between 7 degrees north and 4 degrees south, the answer is $7 + 4 = 11$ degrees, an operation which *appears like addition.*

From these two examples, we learn that a *positive* term is subtracted by changing its sign to *minus*, and a *negative* term is subtracted by changing its sign to *plus.*

This principle is further illustrated by the example below, in which it is plain that the remainders in the lower line must *increase* by 2 throughout, since the numbers to be subtracted *decrease* by 2 throughout; hence, to obtain the true result, the sign of -2 and -4 must be changed to $+$.

3.					
From	16	16	16	16	16
Take	4	2	0	-2	-4
Remainder,	12	14	16	18	20

4. From $+7a$ subtract $+12a$.

OPERATION.

Minuend,	$+7a$
Subtrahend,	$+12a$
Difference,	$-5a$

ANALYSIS. We change the sign of the subtrahend as in the other examples; then $-12a$ and $+7a$, are $-5a$, the algebraic difference.

5.				
From	$15a$	$10a$	$5a$	0
Take	$5a$	$5a$	$5a$	$5a$
Difference,	$10a$	$5a$	0	$-5a$

ANALYSIS. Since the minuends decrease by $5a$ toward the right, and the subtrahends are all equal, the remainders must decrease by $5a$, and the last remainder is therefore $-5a$.

We cannot, numerically, take a greater quantity from a less, nor any quantity from zero, for no quantity can be *less than nothing.* Hence, in the last two examples, the answer,

How is a positive term subtracted? How a negative term? What is understood by a minus quantity?

$-5a$, is not $5a$ less than nothing, but $5a$ applied in the opposite direction to $+5a$. To subtract a quantity algebraically, is *to change the direction in which it is reckoned or applied.* Thus, we see that by a change of sign, we can find the algebraic difference between any two quantities whatever.

From these examples and illustrations we derive the following

RULE. *Change the sign of the subtrahend, or conceive it to be changed, and unite the terms as in addition.*

EXAMPLES FOR PRACTICE.

	(6.)	(7.)	(8.)	(9.)
From	$+4a$	$+6x^2$	$-10bc$	$+4m^2z$
Take	$+a$	$-2x^2$	$-7bc$	$+16m^2z$
Rem.	$+3a$	$+8x^2$	$-3bc$	$-12m^2z$

	(10.)	(11.)	(12.)	(13.)
From	$-16b^2c$	$+13md$	$+27h^2$	$-h^2$
Take	$-17b^2c$	$+15md$	$-h^2$	$+27h^2$
Rem.	$+b^2c$	$-2md$	$+28h^2$	$-28h^2$

14. From $17x^2y$ subtract $-4x^2y$. *Ans.* $21x^2y$.
15. From $abcd$ subtract $-abcd$. *Ans.* $2abcd$.
16. From $25gm$ subtract $28gm$. *Ans.* $-3gm$.
17. From $-16b^2x^2$ subtract $4b^2x^2$. *Ans.* $-20b^2x^2$.
18. From $-11sq^2$ subtract $-12sq^2$. *Ans.* sq^2
19. From $30xy$ subtract $40xy$. *Ans.* $-10xy$.
20. From $75mn^2$ subtract $-25mn^2$. *Ans.* $100mn^2$.
21. From $-75mn^2$ subtract $-25mn^2$. *Ans.* $-50mn^2$.
22. From $-18pqr$ subtract $-17pqr$. *Ans.* $-pqr$.
23. From $14bx^2y$ subtract $17bx^2y$. *Ans.* $-3bx^2y$.
24. From $5(a+b)$ subtract $2(a+b)$. *Ans.* $3(a+b)$.

How may the algebraic difference of any two quantities be found? Give the rule for the subtraction of similar terms

25. From $7a(c-m)$ subtract $-5a(c-m)$.
Ans. $12a(c-m)$.

26. From $-11(x^2-y)$ subtract $-5(x^2-y)$.
Ans. $-6(x^2-y)$.

27. From $12(m-n)$ subtract $-12(m-n)$.
Ans. $24(m-n)$.

28 Subtract $15x^2y^3z$ from $-3x^2y^3z$. *Ans.* $-18x^2y^3z$.

29. Subtract $-157m^3nq$ from $-16m^3nq$. *Ans.* $141m^3nq$.

30. Subtract $172a^2bc$ from $150a^2bc$. *Ans.* $-22a^2bc$.

31. Subtract $7(x^2-y^2-z^2)$ from $12(x^2-y^2-z^2)$.
Ans. $5(x^2-y^2-z^2)$.

32. Subtract $12ab(p-q)$ from $15ab(p-q)$.
Ans. $3ab(p-q)$.

33. Subtract $m^2(c-1)$ from $-2m^2(c-1)$.
Ans. $-3m^2(c-1)$.

CASE II.

53. To find the difference of polynomials.

1. From a subtract $b-c$.

OPERATION.

Minuend,	a
Subtrahend,	$b-c$
Difference,	$a-b+c$

ANALYSIS. We first subtract b from a, and obtain for a result, $a-b$; but our true subtrahend is not b. but $b-c$; and, as we have subtracted a quantity too great by c, our remainder must be too small by c; we therefore add c to the first result, and obtain the true remainder, $a-b+c$.

By this example, we have in a more general manner established the principle, that the sign of a term to be subtracted must be changed. Hence the following

RULE. I. *Write the subtrahend underneath the minuend, placing similar terms under each other.*

II. *Change the signs of the terms of the subtrahend, or conceive them to be changed.*

What is Case II? Give Analysis. Rule.

III. *Unite similar terms as in addition, and bring down all the remaining terms with their proper signs.*

EXAMPLES FOR PRACTICE.

	(2.)	(3.)	(4.)
From	$4a + 2x - 3c$	$3ax + 2y$	$a + b$
Take	$a + 4x - 6c$	$xy - 2y$	$a - b$
Rem	$3a - 2x + 3c$	$3ax - xy + 4y$	$2b$

	(5.)	(6.)	(7.)
From	$2x^2 - 3x + y^2$	$7a + 2 - 5c$	$\frac{1}{2}x + \frac{1}{2}y$
Take	$-x^2 - 4x + a$	$-a + 2 + c$	$\frac{1}{2}x - \frac{1}{2}y$
Rem.	$3x^2 + x + y^2 - a$	$8a \quad - 6c$	y

	(8.)	(9.)
From	$8x^2 - 3xy + 2y^2 + c$	$ab + cd - m^2$
Take	$x^2 - 6xy + 3y^2 - 2c$	$ab - cd - 2m^2$
Rem.	$7x^2 + 3xy - y^2 + 3c$	$2cd + m^2$

	(10.)	(11.)
From	$3x^2 - 2xy + 21a + c$	$3ax - 7by + 4ab$
Take	$-x^2 + 3xy - 4a + 4c$	$-ax - 10by + 2ab$
Rem.		

12. From $8xy - 20$ subtract $-xy + 12$. *Ans.* $9xy - 32$.
13. From $7a^2x + a$ subtract $3a^2x - 2a$. *Ans.* $4a^2x + 3a$.
14. From $-8x - 2y + 3$ subtract $10x - 3y + 4$.
Ans. $-18x + y - 1$.
15. From $6y^2 - 2y - 5$ subtract $-8y^2 - 5y + 12$.
Ans. $14y^2 + 3y - 17$.
16. From $7m^2 - 4ab - c$ subtract $2m^2 + 3c - 8ab - s$.
Ans. $5m^2 + 4ab - 4c + s$.
17. From $a + 2x$ take $a - x$. *Ans.* $3x$

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18. From $4a + 4b$ take $b + a$. *Ans.* $3a + 3b$.

19. From $4a - 4b$ take $3a + 5b$. *Ans.* $a - 9b$.

20. From $13a^2b^3 + 11a - 5a^2 + 6b$, take $7a - 5a^2 + 6b - 10a^2b^3$. *Ans.* $23a^2b^3 + 4a$.

21. From $3a + b + c - d - 10$, take $c + 2a - d$. *Ans.* $a + b - 10$.

22. From $3a + b + c - d - 10$, take $b - 19 + 3a$. *Ans.* $c - d + 9$.

23. From $2ab + b^2 - 4c + bc - b$, take $3a^2 - c + b^2$. *Ans.* $2ab - 3c + bc - 3a^2 - b$.

24. From $a^3 + 3b^2c + ab^2 - abc$, take $b^3 + ab^2 - abc$. *Ans.* $a^3 + 3b^2c - b^3$.

25. From $5x^2y - 3bx + c$, take $3x^2y + 2bx + c^2$. *Ans.* $2x^2y - 5bx - c^2 + c$.

26. From $4m^2 - m + 2cx - y^2$, take $y^2 - 3m^2 - m + cx$. *Ans.* $7m^2 + cx - 2y^2$.

NOTE. — The minus sign before a parenthesis indicates that the whole quantity inclosed is to be subtracted.

27. What is the value of $3a^2 - (3a - x + b)$? *Ans.* $3a^2 - 3a + x - b$.

28. What is the value of $40xy - (30xy - 2b^2 + 3c - 4d)$? *Ans.* $10xy + 2b^2 - 3c + 4d$.

29. What is the value of $a^2 - a - (4a - y - 3a^2 - 1)$? *Ans.* $4a^2 - 5a + y + 1$.

30. What is the value of $7m^2 + 2bc - (3m^2 - bc - x)$? *Ans.* $4m^2 + 3bc + x$.

31. What is the value $a + b - m - (m - a - b)$? *Ans.* $2a + 2b - 2m$.

54. The difference between two dissimilar quantities may often be conveniently expressed in a single term, by making some common letter or letters the unit of subtraction.

Explain the unit of subtraction.

1. From ax subtract bx.

OPERATION.	
Minuend,	ax
Subtrahend,	bx
Remainder,	$(a-b)x$

ANALYSIS. It is evident that b times x taken from a times x must leave a minus b times x, which is expressed thus $(a-b)x$.

EXAMPLES FOR PRACTICE.

	(2.)	(3.)	(4.)	(5.)
From	$2am$	my^2	axy	cx
Take	cm	ny^2	$-cxy$	x
Rem.	$(2a-c)m$	$(m-n)y^2$	$(a+c)xy$	$(c-1)x$

6. From $2abx^2$ take bcx^2. *Ans.* $(2ab-bc)x^2$.
7. From $4xy$ take mxz. *Ans.* $(4y-mz)x$.
8. From $ax+bx+cx$ take $x+ax+bx$. *Ans.* $(c-1)x$.
9. From $3a^2-by$ take $2a^2-cy$. *Ans.* $a^2+(c-b)y$.
10. From $5acx^4+20ax^3y^2-25m$ take $3acx^4+12ax^3y^2-20m$. *Ans.* $2ax^3(cx+4y^2)-5m$.
11. From $(2a+b+c)x$ take $(a+b)x$. *Ans.* $(a+c)x$.
12. From $(3a+c)xy$ take $2axy+cxy$. *Ans.* axy.
13. From $ay+2by-cy$ take $ay+cy$. *Ans.* $(2b-2c)y$.
14. From mz take $nz-5z$. *Ans.* $(m-n+5)z$.
15. From $5a^2x-2x$ take $3x+5ax$. *Ans.* $(5a^2-5a-5)x$.
16. From $-3c^2(m^2-1)$ take $7c^2(m^2-1)$. *Ans.* $-10c^2(m^2-1)$.
17. What is the value of $3cy-x^2y-(my-2x^2y+2cy)$? *Ans.* $(c+x^2-m)y$.
18. What is the value of $3m-z-y-(2z-y-3m)$? *Ans.* $6m-3z$.

MULTIPLICATION.

55. Multiplication, in algebra, is the process of taking one quantity as many times as there are units in another.

CASE I.

56. When both factors are monomials.

1. Multiply $4a$ by $3b$.

OPERATION.

Multiplicand,	$4a$
Multiplier,	$3b$
Product,	$12ab$

ANALYSIS. Since it is immaterial in what order the factors are taken in multiplication, we may proceed thus: 3 times 4 are 12; b times a are ab; and 12 times ab are $12ab$, the entire product.

2. Multiply a^3 by a^2.

OPERATION.

Multiplicand,	a^3
Multiplier,	a^2
Product,	a^5

ANALYSIS. Since a^3 is equal to aaa, and a^2 is equal to aa, their product must be $aaaaa$, which is equal to a^5; this exponent, 5, may be found by adding the two given exponents, 3 and 2.

3. Multiply $3ab^2$ by $4b^3$.

OPERATION.

Multiplicand,	$3ab^2$
Multiplier,	$4\ b^3$
Product,	$12ab^5$

ANALYSIS. 4 times $3a$ are $12a$; and b^3 times b^2 are b^5; hence the entire product is $12a$ times b^5, or $12ab^5$.

57. In the examples given above, all the quantities are understood to be positive. It is necessary, however, to investigate the *law of signs* when one or both the factors are negative. In arithmetic, multiplication is restricted to the simple idea of

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Define Multiplication. What is Case I?

repeating a number. In algebra, quantities have two qualities, and are either *positive* or *negative ;* and multiplication has the double province of repeating by *additions,* and repeating by *subtractions,* as indicated by the *signs of the multiplier.* Hence, the full signification of a multiplier, when analyzed, is as follows:

I. The *plus sign* of a multiplier shows that the multiplicand is to be *added to zero.*

II. The *minus sign* of a multiplier shows that the multiplicand is to be *subtracted from zero ;* and

III. The *value* of the multiplier *shows how many times the multiplicand is to be taken* by either process.

To exhibit the law which governs the sign of a product, according to these principles, we present four examples, as follows:

1. Multiply a by b.

OPERATION.

$$\begin{array}{r} +\,a \\ +\,b \\ \hline +\,ab \end{array}$$

ANALYSIS. The *plus sign* of the multiplier indicates that the multiplicand, $+a$, is to be *added* to zero b times, giving $+a+a+a$, &c.; hence the result will be positive, or $+ab$.

2. Multiply $-a$ by $-b$.

OPERATION.

$$\begin{array}{r} -\,a \\ -\,b \\ +\,ab \end{array}$$

ANALYSIS. The *minus sign* of the multiplier indicates that the multiplicand, $-a$, is to be *subtracted* from zero b times, which will change its sign, giving $+a+a+a$, &c.; hence the result will be *positive,* or $+ab$.

3. Multiply a by $-b$.

OPERATION.

$$\begin{array}{r} +\,a \\ -\,b \\ \hline -\,ab \end{array}$$

ANALYSIS. The *minus sign* of the multiplier indicates that the multiplicand, $+a$, is to be *subtracted* from zero b times, which will change its sign, giving $-a-a-a$, &c.; hence, the result will be *negative,* or $-ab$.

Explain the difference between Arithmetical and Algebraic multiplication. How do the signs + and — before a multiplier affect the product? The value of a multiplier shows what?

4. Multiply $-a$ by b.

OPERATION.	
$-a$	ANALYSIS. The *plus sign* of the multiplier indicates that the multiplicand, $-a$, is to be *added* to zero b times, which only repeats the letter, giving $-a-a-a$, &c.; hence the result will be *negative*, or $-ab$.
$+b$	
$-ab$	

Comparing the four examples, we observe that *like signs produce plus; and unlike, minus.*

From the foregoing examples and illustrations we derive the following

RULE. I. *Multiply the coefficients of the two terms together for the coefficient of the product.*

II. *Write all the letters of both terms for the literal part, giving each letter an exponent equal to the sum of its exponents in the two terms.*

III. *If the signs of the two terms are alike, make the product plus; if unlike, make it minus.*

NOTE 1. The value of the product will be the same in whatever order the factors are written. Most algebraists prefer to arrange letters in alphabetical order.

EXAMPLES FOR PRACTICE.

5. Multiply $3x$ by $7a$. *Ans.* $21ax$.
6. Multiply $4y$ by $3ab$. *Ans.* $12aby$.
7. Multiply $15bc$ by $10x$. *Ans.* $150bcx$.
8. Multiply $6ax$ by $12by$. *Ans.* $72abxy$.
9. Multiply $17cd$ by $3m$. *Ans.* $51cdm$.
10. Multiply $4pq$ by $7xy$. *Ans.* $28pqxy$.
11. Multiply $12am$ by $5bcd$. *Ans.* $60abcdm$.
12. Multiply $25pqr$ by $3xyz$. *Ans.* $75pqrxyz$.
13. What is the product of a^3 by a^5? *Ans.* a^8.
14. What is the product of x^4 by x^6? *Ans.* x^{10}.
15. What is the product of y^5 by y^5? *Ans.* y^{10}.

If the factors have like signs, what must be the sign of the product? If unlike, what? Give the rule for multiplication of monomials.

16. What is the product of m^7 by m^5? *Ans.* m^{12}.
17. What is the product of b^4x^3 by b^3x^5? *Ans.* b^7x^8.
18. What is the product of a^2m^5 by am^2? *Ans.* a^3m^7.
19. Multiply $4ac$ by $-3ab$. *Ans.* $-12\,a^2bc$.
20. Multiply $9a^2c$ by $-4ay$. *Ans.* $-36a^3cy$.
21. Multiply $-2xy$ by $-2xy$. *Ans.* $4x^2y^2$.
22. Multiply $-7ay$ by $3xy$. *Ans.* $-21axy^2$.
23. Multiply $21x^2y$ by $-3xy$. *Ans.* $-63x^3y^2$.
24. Multiply $-5a^2m$ by $-4abm^2$. *Ans.* $20a^3bm^3$.
25 Multiply $-7m^2$ by $10c^2m^3z$. *Ans.* $-70c^2m^5z$.
26. Multiply $17x^2y^3$ by $2x^3y^2$. *Ans.* $34x^5y^5$.
27. Multiply $14ab^2cd^5$ by $-3b^3c^2m$.
Ans. $-42ab^5c^3d^5m$.
28. What is the value of $3a \times 4b \times 2c$? *Ans.* $24abc$.
29. What is the value of $7m^2 \times 4am \times 2my$?
Ans. $56am^4y$.
30. What is the value of $x^3 \times x^3 \times x^3$? *Ans.* x^9.
31. What is the value of $-7a^2b \times 2ab^2 \times 3ab$?
Ans. $-42a^4b^4$.
32. What is the value of $-5a^2m \times 3ab^2c \times 2bc^2m^2$?
Ans. $-30a^3b^3c^3m^3$.
33. Multiply $3(x + y)$ by 2. *Ans.* $6(x + y)$.
34. Multiply $a(x^2 + m)$ by b. *Ans.* $ab(x^2 + m)$.
35. Multiply $(a + m)^2$ by c. *Ans.* $c(a + m)^2$.
36. Multiply $(a + b)^3$ by $(a + b)^2$. *Ans.* $(a + b)^5$.
37. Multiply $3a(m - n)^2$ by $-a(m - n)^3$.
Ans. $-3a^2(m - n)^5$.
38. Multiply $4m(x^2 - y^2)^3$ by $-2am(x^2 - y^2)$.
Ans. $-8am^2(x^2 - y^2)^4$.

Note.—When quantities have literal exponents, powers of the same letter or quantity are multiplied by indicating the addition of the exponents.

39. Multiply a^m by a^n. *Ans.* a^{m+n}.
40 Multiply c^m by c. *Ans.* c^{m+1}.

How are quantities with literal exponents multiplied?

41. Multiply $(a - b)^3$ by $(a - b)^2$.

Ans. $(a - b)^{3+2}$.

42 Multiply $a^m(p + q)^2$ by $a^2(p + q)^m$.

Ans. $a^{m+2}(p + q)^{2+m}$.

CASE II.

58. When one factor is a polynomial.

1. Multiply $4b + 5a^2 - bc$ by $3a$.

OPERATION.

$$\begin{array}{l} 4b + 5a^2 - bc \\ 3a \\ \hline 12ab + 15a^3 - 3abc \end{array}$$

ANALYSIS. Since the whole multiplicand is to be taken $3a$ times, we must multiply each of its terms by $3a$; thus $3a$ times $4b$ is $12ab$; $3a$ times $5a^2$ is $15a^3$; $3a$ times $-bc$ is $-3abc$; and we have for the entire product, $12ab + 15a^3 - 3abc$. Hence the

RULE. *Multiply each term of the polynomial separately by the multiplier, and write the partial products connected by their proper signs.*

EXAMPLES FOR PRACTICE.

(2.)	(3.)	(4.)
$\begin{array}{l} 5a - 3c \\ 2a \\ \hline 10a^2 - 6ac \end{array}$	$\begin{array}{l} 3ac - 4b \\ -3a \\ \hline -9a^2c + 12ab \end{array}$	$\begin{array}{l} 2a^2 - 3c + 5 \\ bc \\ \hline 2a^2bc - 3bc^2 + 5bc \end{array}$

(5.)	(6.)	(7.)
$\begin{array}{l} 12x - 2ac \\ 4a \\ \hline \end{array}$	$\begin{array}{l} 15c - 7b \\ -2a \\ \hline \end{array}$	$\begin{array}{l} 4x - b + 3ab \\ 2ab \\ \hline \end{array}$

(8.)	(9.)	(10.)
$\begin{array}{l} 3c^2 + x \\ 4xy \\ \hline \end{array}$	$\begin{array}{l} 10x^3 - 3y^3 \\ -4x^2 \\ \hline \end{array}$	$\begin{array}{l} 3a^2 - 2x^3 - 6b \\ 2ax^3 \\ \hline \end{array}$

Give Case II. Analysis. Rule.

11. Multiply $3b - 2c$ by $5bc$ *Ans.* $15b^2c - 10bc^2$.

12. Multiply $4xy - 9$ by $6x$. *Ans.* $24x^2y - 54x$.

13. Multiply $a^2 - 2x + 1$ by $4x^2$.
Ans. $4a^2x^2 - 8x^3 + 4x^2$

14. Multiply $11a^3bc^2 - 13xy$ by $3ax$.
Ans. $33a^4bc^2x - 39ax^2y$.

15. Multiply $42c^2 - 1$ by -4. *Ans.* $-168c^2 + 4$.

16. Multiply $-30a^2bx^2y + 13$ by $-5a^3$.
Ans. $150a^5bx^2y - 65a^3$.

17. Multiply $2b - 7a - 3$ by $4ab$.
Ans. $8ab^2 - 28a^2b - 12ab$.

18. Multiply $a + 3b - 2c$ by $-3ab$.
Ans. $-3a^2b - 9ab^2 + 6abc$.

19. Multiply $13a^2 - b^2c$ by $-4c$.
Ans. $-52a^2c + 4b^2c^2$.

20. Multiply $13xy - 3b$ by $-25x^2$.
Ans. $-325x^3y + 75bx^2$.

CASE III.

59. When both factors are polynomials.

1. Multiply $2a + 3b$ by $a + b$.

OPERATION.

Multiplicand,	$2a + 3b$
Multiplier,	$a + b$
Product by a,	$2a^2 + 3ab$
Product by b,	$+ 2ab + 3b^2$
Entire Product,	$2a^2 + 5ab + 3b^2$

ANALYSIS. To multiply by $a + b$, we must take the multiplicand a times and b times, which is done by multiplying by a and b separately, and adding the partial products. Hence the

RULE. *Multiply all the terms of the multiplicand by each term of the multiplier separately, and add the partial products.*

Give Case III. Analysis. Rule.

EXAMPLES FOR PRACTICE.

(2.)

$$\begin{array}{lr}
\text{Multiply} & 2a + 5c \\
 & a - c \\
\hline
 & 2a^2 + 5ac \qquad\quad \\
 & -2ac - 5c^2 \\
\hline
\text{Product,} & 2a^2 + 3ac - 5c^2
\end{array}$$

(3.)

$$\begin{array}{lr}
\text{Multiply} & 3x - 5y \\
 & x - 2y \\
\hline
 & 3x^2 - 5xy \qquad\quad \\
 & -6xy + 10y^2 \\
\hline
\text{Product,} & 3x^2 - 11xy + 10y^2
\end{array}$$

(4.)

$$\begin{array}{ll}
\text{Multiply} & 6x - 2z \\
\text{By} & 3a - 5d \\
\hline
\text{Product,} & 18ax - 6az - 30dx + 10dz
\end{array}$$

(5.)

$$\begin{array}{ll}
\text{Multiply} & a + b + c \\
\text{By} & x + y + z \\
\hline
\text{Product,} & ax + bx + cx + ay + by + cy + az + bz + cz.
\end{array}$$

6. Multiply $3a^2 - 2ab - b^2$ by $2a - 4b$.
Ans. $6a^3 - 16a^2b + 6ab^2 + 4b^3$.

7. Multiply $x^2 - xy + y^2$ by $x + y$. *Ans.* $x^3 + y^3$.

8. Multiply $3a + 4c$ by $2a - 5c$.
Ans. $6a^2 - 7ac - 20c^2$.

9. Multiply $a^2 + ay - y^2$ by $a - y$.
Ans. $a^3 - 2ay^2 + y^3$.

10. Multiply $a^2 + ay + y^2$ by $a - y$. *Ans.* $a^3 - y^3$.

11. Multiply $a^2 - ay + y^2$ by $a + y$. *Ans.* $a^3 + y^3$.

12. Multiply $a^3 + a^2y + ay^2 + y^3$ by $a - y$.
Ans. $a^4 - y^4$.

13 Multiply $y^2 - y + 1$ by $y + 1$. *Ans.* $y^3 + 1$.

14. Multiply $x^2 + y^2$ by $x^2 - y^2$. *Ans.* $x^4 - y^4$.

15. Multiply $a^2 - 3a + 8$ by $a + 3$. *Ans.* $a^3 - a + 24$.

16. Multiply $b^4 + b^2x^2 + x^4$ by $b^2 - x^2$. *Ans.* $b^6 - x^6$.

17. Multiply $a^2 + 2b$ by $2a^2 - 4b$. *Ans.* $2a^4 - 8b^2$.

18. Multiply $x^6 + x^4 + x^2$ by $x^2 - 1$. *Ans.* $x^8 - x^2$.

19. Multiply $m + n$ by $9m - 9n$. *Ans.* $9m^2 - 9n^2$.

20. Multiply $2x^2 + xy - 2y^2$ by $3x - 3y$.
Ans. $6x^3 - 3x^2y - 9xy^2 + 6y^3$.

21. Multiply $m^2 - 3m - 7$ by $m - 2$.
Ans. $m^3 - 5m^2 - m + 14$.

22. Multiply $a^4 - 2a^3c + 4a^2c^2 - 8ac^3 + 16c^4$ by $a + 2c$.

23. Multiply $x^3 - 3x^2 + 3x - 9$ by $x + 3$.
Ans. $x^4 - 6x^2 - 27$.

24. Multiply $m^4 - m^3 + m^2 - m + 1$ by $m + 1$.
Ans. $m^5 + 1$.

25. Multiply $m^4 + m^3 + m^2 + m + 1$ by $m - 1$.

26. Multiply $2a^3 + 5ac^2 - 2c^3$ by $2a^3 - 5ac^2 + 2c^3$.
Ans. $4a^6 - 25a^2c^4 + 20ac^5 - 4c^6$.

NOTE. — The product of two or more polynomials may be indicated by inclosing each in a parenthesis, and writing them in succession; such an expression is said to be *expanded*, when the multiplication has been actually performed.

27. Expand $(a + b)(a + c)$.
Ans. $a^2 + ab + ac + bc$.

28. Expand $(x + 3y)(x^2 - y)$.
Ans. $x^3 + 3x^2y - xy - 3y^2$.

29. Expand $(m^2 + 2c)(m^2 - 5c)$.

30. Expand $(a + b - c)(a - b + c)$.
Ans. $a^2 - b^2 + 2bc - c^2$.

31. Expand $(a - c - 1)(a + 1)$.
Ans. $a^2 - ac - c - 1$.

CASE IV.

60. To square a binomial.

If a polynomial be multiplied by itself, the product is the *square* of the polynomial. A binomial quantity is easily

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How may multiplication of polynomials be indicated? When are such expressions expanded? Give Case IV.

squared without the formal process of multiplying, as will be seen by the analyses of the two following examples.

1. What is the square of $a + b$?

OPERATION.

$$\begin{array}{l} a + b \\ \underline{a + b} \\ a^2 + ab \\ \underline{\quad + ab + b^2} \\ a^2 + 2ab + b^2 \end{array}$$

ANALYSIS. Multiplying $a + b$ by $a + b$ by the common method, we obtain for the result, a^2, which is the square of a; $+ 2ab$, which is the 2 times product of a and b; and b^2, which is the square of b.

2. What is the square of $a - b$?

OPERATION.

$$\begin{array}{l} a - b \\ \underline{a - b} \\ a^2 - ab \\ \underline{\quad - ab + b^2} \\ a^2 - 2ab + b^2 \end{array}$$

ANALYSIS. Multiplying in the usual way, we obtain for the result a^2, which is the square of a; $-2ab$, which is twice the product of a and $-b$; and b^2, which is the square of b. Hence the

RULE. *Write the square of the first term, twice the product of the two terms, and the square of the second term.*

NOTE.—The product of the two terms will be minus, when one of them is minus.

EXAMPLES FOR PRACTICE.

3. Square $a + c$. *Ans.* $a^2 + 2ac + c^2$.
4. Square $p + q$. *Ans.* $p^2 + 2pq + q^2$.
5. Square $m - n$. *Ans.* $m^2 - 2mn + n^2$.
6 Square $x - y$. *Ans.* $x^2 - 2xy + y^2$.
7 Square $A + B$. *Ans.* $A^2 + 2AB + B^2$.
8. Square $A - C$. *Ans.* $A^2 - 2AC + C^2$.
9. Square $3a - 2x$. *Ans.* $9a^2 - 12ax + 4x^2$.

Give analysis. Rule.

10. Expand $(m+z)(m+z)$. *Ans.* $m^2+2mz+z^2$.
11. Expand $(2a-c)(2a-c)$. *Ans.* $4a^2-4ac+c^2$.
12. Expand $(5x-3)(5x-3)$. *Ans.* $25x^2-30x+9$
13. Expand $(4a+\frac{1}{2}x)(4a+\frac{1}{2}x)$.
Ans. $16a^2+4ax+\frac{1}{4}x^2$.

Note.—The square of a binomial may be indicated by an exponent.

14. Expand $(m+c)^2$. *Ans.* $m^2+2cm+c^2$.
15. Expand $(2c-3d)^2$. *Ans.* $4c^2-12cd+9d^2$.
16. Expand $(x^2-x)^2$. *Ans.* $x^4-2x^3+x^2$.
17. Expand $(a-1)^2$. *Ans.* a^2-2a+1.
18. Expand $(a^2x-ax^2)^2$. *Ans.* $a^4x^2-2a^3x^3+a^2x^4$.
19. Expand $(y^2-20)(y^2-20)$.
Ans. y^4-40y^2+400.
20. Expand $(x^m-y^n)(x^m-y^n)$.
Ans. $x^{2m}-2x^my^n+y^{2n}$.
21. Expand $(c^m-1)(c^m-1)$. *Ans.* $c^{2m}-2c^m+1$.

CASE V.

61. To find the product of the sum and difference of two quantities.

The *sum* of two quantities multiplied by their *difference* gives a result still more simple than a binomial square.

1. Multiply $a+b$ by $a-b$.

OPERATION.

$$\begin{array}{lr}
\text{Sum,} & a+b \\
\text{Difference,} & a-b \\
\hline
 & a^2+ab \\
 & -ab-b^2 \\
\hline
\text{Product,} & a^2-b^2
\end{array}$$

Analysis. We are required to multiply the *sum* of a and b, by the *difference* of a and b. Multiplying by the usual process, we find in adding the partial products, that $+ab$ and $-ab$ reduce to zero, and the product is a^2-b^2, or the *difference of the squares* of a and b. But since a and b may represent any two quantities whatever, the form of this product embodies a *general truth*. Hence

Give Case V. Analysis.

Rule. *From the square of the greater quantity, subtract the square of the less.*

Note.—The term or quantity having the minus sign in the difference, is supposed to be the less.

EXAMPLES FOR PRACTICE.

2. What is the product of $m+n$ by $m-n$? *Ans.* m^2-n^2.

3. What is the product of $a+c$ by $a-c$? *Ans.* a^2-c^2.

4. What is the product of $A+B$ by $A-B$? *Ans.* A^2-B^2.

5. What is the product of $2m+2n$ by $2m-2n$? *Ans.* $4m^2-4n^2$.

6. What is the product of $x+y$ by $x-y$? *Ans.* x^2-y^2.

7. What is the product of $3x+3y$ by $3x-3y$? *Ans.* $9x^2-9y^2$.

8. What is the product of $7a+b$ by $7a-b$? *Ans.* $49a^2-b^2$.

9. What is the product of $1+10a$ by $1-10a$? *Ans.* $1-100a^2$.

10. Expand $(1-c^m)(1+c^m)$. *Ans.* $1-c^{2m}$.

11. Expand $(1+2x)(1-2x)$. *Ans.* $1-4x^2$.

12. Expand $(a+\frac{1}{2}x)(a-\frac{1}{2}x)$. *Ans.* $a^2-\frac{1}{4}x^2$.

13. Expand $(\overline{x+y}+z)(\overline{x+y}-z)$. *Ans.* $(x+y)^2-z^2$.

14. Expand $(1+\overline{m-c})(1-\overline{m-c})$. *Ans.* $1-(m-c)^2$.

15. Multiply $(a+b)+(x+y)$ by $(a+b)-(x+y)$ *Ans.* $(a+b)^2-(x+y)^2$.

Note.—To simplify example 15, let $P=a+b$, and $Q=x+y$, then the product required will be $(P+Q)(P-Q)=P^2-Q^2$.

Give Rule.

16. What is the product of $\overline{a + 2xy} \times \overline{a - 2xy} \times 5b$?

Ans. $5a^2b - 20bx^2y^2$.

17. What is the product of $(2A^m + 3B^nC^x)\ (2A^m - 3B^nC^x)$?

Ans. $4A^{2m} - 9B^{2n}C^{2x}$.

62. By applying the principles established in the last two articles, much labor may often be saved when it is required to find the product of three or more binomials.

1. What is the value of $(x + 3)\ (x + 3)\ (x + 3)$ when expanded?

OPERATION.

$$\begin{array}{rrrr} & x^2 + 6x & + 9 & \\ & x + 3 & & \\ \hline & x^3 + 6x^2 & + \ 9x & \\ & 3x^2 & + 18x & + 27 \\ \hline \text{Product,} & x^3 + 9x^2 & + 27x & + 27 \end{array}$$

ANALYSIS. We are able at once to *write* the product of the first two factors, $(x + 3)\ (x + 3)$, which is $x^2 + 6x + 9$, (**60**); and multiplying by $x + 3$, the other factor, we obtain $x^3 + 9x^2 + 27x + 27$, the final product.

2. Expand $(a + b)\ (a - b)\ (a - c)$.

OPERATION.

$$\begin{array}{rl} (a + b)\ (a - b) = & a^2 - b^2 \\ & a - c \\ \hline \text{Product,} & a^3 - ab^2 - a^2c + b^2c. \end{array}$$

ANALYSIS. We expand by the rule (**61**), $(a + b)\ (a - b)$, and then multiply the result by $a - c$.

3. Expand, $(x - c)\ (x - d)\ (x + c)\ (x + d)$.

OPERATION.

$$\begin{array}{l} x^2 - c^2 \\ x^2 - d^2 \\ \hline x^4 - c^2x^2 - d^2x^2 + c^2d^2 \end{array}$$

ANALYSIS. We write $x^2 - c^2$, the product of the first and third factors; and under this $x^2 - d^2$, the product of the second and fourth factors; then we multiply these two products together, and obtain the product of the four factors.

NOTE. — In obtaining the continued product of several factors, the pupil should use judgment in selecting and combining the factors, so as to enable him as far as possible to write out the results mentally.

EXAMPLES FOR PRACTICE.

4. Expand $c(m-n)(m+n)$. *Ans.* cm^2-cn^2.

5. Expand $(3a-b)(3a-b)x$.
Ans. $9a^2x-6abx+b^2x$.

6. Expand $(2m-c)(2m+c)(4m^2+c^2)$.
Ans. $16m^4-c^4$.

7. Expand $(a+c)(a+d)(a-c)(a-d)$.
Ans. $a^4-a^2c^2-a^2d^2+c^2d^2$.

8. Expand $(1+c)(1+c)(1-c)(1+c^2)$.
Ans. $1+c-c^4-c^5$.

9. Expand $(x-4)(x-5)(x+4)(x+5)$.
Ans. x^4-41x^2+400.

10. Expand $(3x-m)(x^2+m^2)(3x-m)$.
Ans. $9x^4-6x^3m+10x^2m^2-6xm^3+m^4$.

11. Expand $(2a+3x)(2a+3x)9$.
Ans. $36a^2+108ax+81x^2$.

12. Expand $(7cd^2+4y^3z)(7cd^2-4y^3z)$.
Ans. $49c^2d^4-16y^6z^2$.

13. Expand $(x+1)(x+1)(x-2)$.
Ans. x^3-3x-2.

14. Expand $(m-2)(m-2)(m+1)$.
Ans. m^3-3m^2+4.

15. Expand $(m^{16}+1)(m^8+1)(m^4+1)(m^2+1)(m+1)(m-1)$. *Ans.* $m^{32}-1$.

DIVISION.

63. **Division,** in Algebra, is the process of finding how many times one quantity is contained in another. It is the converse of multiplication, the dividend answering to the product, and the divisor and quotient to the multiplier and multiplicand.

CASE I.

64. When both dividend and divisor are monomials.

1. Divide $6ab$ by $2a$.

OPERATION.

$$6ab \div 2a = 3b$$

ANALYSIS. Since division is the converse of multiplication, we must seek for a quantity, which multiplied by $2a$. the divisor, will produce $6ab$, the dividend. This quantity is $3b$, and is found by *inspection*, or by dividing 6, the coefficient of the dividend, by 2, the coefficient of the divisor, and dropping the factor a, common to both dividend and divisor.

2. Divide x^5 by x^2.

OPERATION.

$$x^5 \div x^2 = x^3$$

ANALYSIS. Since the divisor multiplied by the quotient will produce the dividend, the *exponent* of the quotient must be such as, when *added* to the exponent of the divisor, will be equal to that of the dividend. Hence, we subtract 2, the exponent of the divisor, from 5, the exponent of the dividend, and obtain 3, the exponent of the quotient.

NOTE. — To *multiply* one power by another of the same letter, we *add* exponents; to *divide* one power by another of the same letter, we *subtract* exponents.

3. Divide $a^2m^5z^3$ by m^2z^3.

OPERATION.

$$a^2m^5z^3 \div m^2z^3 = a^2m^3$$

ANALYSIS. The exponent of a in the quotient is 2; the exponent of m is $5 - 2 = 3$; and the exponent of z is $3 - 3 = 0$; this signifies that z is taken *no* times in the quotient, and is therefore cancelled.

Define Division. Show its relation to Multiplication.

65. In the foregoing examples, no signs being expressed, both dividend and divisor are understood to be positive. To ascertain what sign the quotient should have when one or both the terms of division have the minus sign, we have only to observe what sign must be given to the quotient, in order that the product of the quotient and divisor shall have the same sign as the dividend, according to the law of signs in multiplication.

To exhibit the law which governs the sign of the quotient, we present four examples.

1. $+ab \div +a = +b$ because $+a \times +b = +ab.$
2. $-ab \div -a = +b$ " $-a \times +b = -ab.$
3. $+ab \div -a = -b$ " $-a \times -b = +ab.$
4. $-ab \div +a = -b$ " $+a \times -b = -ab.$

From these examples, taken in order, we make the following inferences:

$$\left.\begin{array}{l} 1st.\ + \text{ divided by } + \text{ gives } + \\ 2d.\ - \text{ divided by } - \text{ gives } + \end{array}\right\} \text{or, like signs produce } +$$

$$\left.\begin{array}{l} 3d.\ + \text{ divided by } - \text{ gives } - \\ 4th.\ - \text{ divided by } + \text{ gives } - \end{array}\right\} \text{or, unlike signs produce } -$$

66. These principles may be deduced from the nature of the signs themselves, by taking another view of division.

Division, considered in its most elementary sense, is not merely the *converse of multiplication;* it is a short process of finding how many times one quantity can be subtracted from another of the same kind. When the *subtraction is possible,* and diminishes the numeral value of the minuend, and brings it nearer to zero, the operation is *real* and must be marked *plus.* When the *subtraction is not possible* without going farther from zero, we must take the *converse* operation, and the converse operation we must mark *minus.*

Give analyses of the law which governs the signs of the quotient. Give the law.

Thus, divide $18a$ by $6a$. Here, it is proposed to find how many times $6a$ can be *subtracted* from $18a$; and as we can *actually* subtract it 3 times, the quotient must be $+ 3$.

Divide $-18a$ by $-6a$. Here, again, the subtraction can *actually be performed*, and the number of times is 3, and, of course the quotient is $+ 3$.

Divide $-18a$ by $6a$. Here, subtraction will not reduce the dividend to zero; but *addition* will, and must be performed 3 times; but the operation is the converse of the one proposed, and therefore must be marked by the converse sign to *plus*, that is -3.

Again, divide $18a$ by $-6a$. Here, if we *subtract* $-6a$ it will not reduce $18a$; but the converse operation will, and therefore the quotient must be minus, that is, -3.

From all these illustrations we derive the following:

RULE. I. *Divide the coefficient of the dividend by the coefficient of the divisor, for a new coefficient.*

II. *Write the letters of the dividend in the quotient, giving each an exponent equal to the difference of its exponents in the two terms, and suppressing all letters whose exponents become zero.*

III. *If the signs of the terms are alike, make the quotient plus; if unlike, make it minus.*

NOTE.—If the dividend does not exactly contain the divisor, the division may be indicated by writing the dividend above a horizontal line, and the divisor below, in the form of a fraction; and the result thus obtained may be simplified by canceling all the factors common to the two terms; thus, $4a^2b^5c \div 6a^2b^4c^2 = \frac{4a^2b^5c}{6a^2b^4c^2} = \frac{2b}{3c}$. But this process is essentially a case of reduction of fractions; we shall therefore omit all examples of this class till the section on fractions is reached.

EXAMPLES FOR PRACTICE.

1. Divide $16ab$ by $4a$. *Ans.* $4b$.
2. Divide $21acd$ by $7c$. *Ans.* $3ad$.
3. Divide ab^2c by ac. *Ans.* b^2.

Rule for division of monomials.

4. Divide $6abc$ by $2c$. *Ans.* $3ab$.
5. Divide ax^3 by ax^2. *Ans.* x.
6. Divide $3mx^6$ by mx. *Ans.* $3x^5$.
7. Divide $210c^3b$ by $7cb$. *Ans.* $30c^2$.
8. Divide $42xy$ by xy. *Ans.* 42.
9. Divide $-21ac$ by $-7a$. *Ans.* $3c$.
10. Divide $-12xy$ by $3y$. *Ans.* $-4x$.
11. Divide $72abc$ by $-8c$. *Ans.* $-9ab$.
12. Divide $2a^6$ by a^4. *Ans.* $2a^2$.
13. Divide $-a^7$ by a^6. *Ans.* $-a$.
14. Divide $16x^3$ by $4x$. *Ans.* $4x^2$.
15. Divide $15axy^3$ by $-3ay$. *Ans.* $-5xy^2$.
16. Divide $-18a^2x$ by $-6ax$. *Ans.* $3a^2$.
17. Divide $6acdxy^2$ by $2adxy^2$ *Ans.* $3c$.
18. Divide $12a^2x^2$ by $-3a^2x$. *Ans.* $-4x$.
19. Divide $15ay^2$ by $-3ay$. *Ans.* $-5y$.
20. Divide $45(a-x)^3$ by $15(a-x)^2$. *Ans.* $3(a-x)$.

Note.—In this example, consider $(a-x)$ as one quantity.

21. Divide $45y^3$ by $15y^2$. *Ans.* $3y$.

Note.—Examples 20 and 21 are exactly alike, if we conceive $(a-x)$ equal to y.

22. Divide z^5 by z^3. *Ans.* z^2.
23. Divide $(x-y)^5$ by $(x-y)^3$. *Ans.* $(x-y)^2$.

Note.—Observe that examples 22 and 23 are essentially alike.

24. Divide $(a+b)^4$ by $(a+b)$. *Ans.* $(a+b)^3$.
25. Divide x^m by x^n. *Ans.* x^{m-n}.
26. Divide $6c^m$ by $3c$. *Ans.* $2c^{m-1}$.
27. Divide $(a-c)^m$ by $(a-c)^2$. *Ans.* $(a-c)^{m-2}$.
28. Divide $10(a-c)$ by $5(a-c)$. *Ans.* 2.
29. Divide $6a^2(a+m)^4$ by $2a(a+m)$. *Ans.* $3a(a+m)^3$.
30. Divide $52m^2c(1-x^2)^4$ by $13mc(1-x^2)^4$. *Ans.* $4m$.
31. Divide $81a^4z^2(4m-q)^5$ by $27z^2(4m-q)^4$ *Ans.* $3a^4(4m-q)$.

CASE II.

67. To divide a polynomial by a single term.

1. Divide $12a^5 - 6a^3c + 3a^2m$ by $3a^2$.

OPERATION.

$$3a^2)\underline{12a^5 - 6a^3c + 3a^2m}$$
$$4a^3 - 2ac + m$$

ANALYSIS. The whole dividend is divided by $3a^2$, by dividing each of its terms by $3a^2$.

Hence the

RULE. *Divide each term of the dividend separately, and connect the quotients by their proper signs.*

EXAMPLES FOR PRACTICE

2. Divide $15ab - 12ax$ by $3a$. *Ans.* $5b - 4x$.
3. Divide $-25a^2x + 15ax^2$ by $-5ax$. *Ans.* $5a - 3x$.
4. Divide $10ab + 15ac$ by $5a$. *Ans.* $2b + 3c$.
5. Divide $30ax - 54x$ by $6x$. *Ans.* $5a - 9$.
6. Divide $8x^3 + 12x$ by $4x^2$. *Ans.* $2x + 3x^{-1}$.
7. Divide $3bcd + 12bcx - 9b^2c$ by $3bc$.
   Ans. $d + 4x - 3b$.
8. Divide $7ax + 7ay - 7ad$ by $-7a$.
   Ans. $-x - y + d$.
9. Divide $3ax^3 + 6x^2 + 3ax - 15x$ by $3x$.
   Ans. $ax^2 + 2x + a - 5$.
10. Divide $3ab^4c + 12ab^5x - 3a^2b^7$ by $3ab^3$.
    Ans. $bc + 4b^2x - ab^4$.
11. Divide $25a^2bx - 15a^2cx^2 + 5a^3bcx^2$ by $-5a^2x$.
    Ans. $-5b + 3cx - abcx$.
12. Divide $20a^3b^3 + 15a^3b^2 + 10a^3b + 5a$ by $5a^3$.
    Ans. $4b^3 + 3b^2 + 2b + a^{-2}$.
13. Divide $21a + 35b - 14$ by -7.
    Ans. $2 - 3a - 5b$.

Give Case II. Analysis. Rule.

14. Divide $-12a^2bc + 9acx^2 - 6ab^2c$ by $-3ac$.
Ans. $4ab - 3x^2 + 2b^2$.

15. Divide $6(a + x) + 9(x + y)$ by 3.
Ans. $2(a + x) + 3(x + y)$.

16. Divide $12(a + x) - 3c(a + x) + d(a + x)$ by $(a + x)$.
Ans. $12 - 3c + d$.

17. Divide $(a + c)^2 - (a + c)^5$ by $(a + c)$.
Ans. $(a + c) - (a + c)^4$.

18. Divide $12(a - b) + 6c(a - b) + 2(a - b)$ by $(a - b)$.
Ans. $12 + 6c + 2$.

19. Divide $(m + n)x^2 + (m + n)a^2 + (m + n)c^2$ by $(m + n)$.
Ans. $x^2 + a^2 + c^2$.

20. Divide $(a + b)^2 + 2(a + b)$ by $(a + b)$.
Ans. $a + b + 2$.

NOTE.—When a parenthesis has the unit 1 for both coefficient and exponent, and is connected with the other parts of the algebraic expression by $+$ or $-$, it may be omitted; thus, $(a + b) + 2$, is the same as $a + b + 2$. But when a parenthesis having the minus sign before it is dropped, the signs of the quantities inclosed must all be changed (58); thus, $a^2 - (a - x)$, is the same as $a^2 - a + x$

21. Divide $2a(a + c) + (a + c)^2$ by $(a + c)$.
Ans. $3a + c$.

22. Divide $5c(3m - 2c) - (3m - 2c)^2$ by $(3m - 2c)$.
Ans. $7c - 3m$.

23. Divide $(1 - x) - (1 - x)^2$ by $(1 - x)$. *Ans.* x.

CASE III.

68. To divide one polynomial by another.

Since the dividend is always the *product* of the divisor by the quotient sought, the highest power of any letter of the dividend must be the product of the highest powers of the same letter in the divisor and quotient; and the inferior powers of this letter in the dividend, must be the products of inferior powers in divisor and quotient. Hence *the terms of both*

Give Case III.

divisor and dividend must be arranged in the order of the powers of one of the letters.

1. Divide $2a^4 + 5a^2b^2 + 2a^3b - 6ab^3 + 4b^4$ by $a^2 + 2ab + 4b^2$.

OPERATION.

$$
\begin{array}{lr|l}
\text{Dividend,} & 2a^4 + 2a^3b + 5a^2b^2 - 6ab^3 + 4b^4 & a^2 + 2ab + 4b^2, \text{ Divisor.} \\
 & \underline{2a^4 + 4a^3b + 8a^2b^2 \qquad\qquad\quad} & \overline{2a^2 - 2ab + b^2, \text{ Quotient.}} \\
\text{1st Rem.} & - 2a^3b - 3a^2b^2 - 6ab^3 \qquad & \\
 & \underline{- 2a^3b - 4a^2b^2 - 8ab^3 \qquad} & \\
\text{2d Rem.} & a^2b^2 + 2ab^3 + 4b^4 & \\
 & \underline{a^2b^2 + 2ab^3 + 4b^4} &
\end{array}
$$

ANALYSIS. We arrange the terms of both divisor and dividend according to the powers of a, so that in the dividend the exponents of this letter, taken in their order, are 4, 3, 2, 1; and in the divisor, 2, 1. Now, according to the principle just stated, the first term of the dividend, thus arranged, must be equal to the first term of the divisor multiplied by that term of the quotient having the highest power of a; we therefore divide $2a^4$, the first term of the dividend, by a^2, the first term of the divisor, and obtain $2a^2$ for the first of the quotient. We next multiply the *whole divisor* by this term of the quotient, and subtract the product from the dividend, bringing down as many terms as are necessary for a new dividend. We then divide $-2a^3b$, the first term of the remainder, by a^2, the first term of the divisor, and obtain $-2ab$ for the second term of the quotient. We next multiply the *whole divisor* by this term of the quotient, and subtract the product from the second dividend, and obtain a second remainder to which we annex another term of the dividend for another dividend. Dividing a^2b^2, the first term of this dividend, by a^2, the first term of the divisor, we obtain b^2, another term of the quotient. Lastly, multiplying the whole divisor by this term of the quotient, and subtracting the product from the last dividend, we have no remainder, and the work is finished.

From this example we derive the following

RULE. I. *Arrange both divisor and dividend with reference to the powers of one of the letters.*

II. *Divide the first term of the dividend by the first term of the divisor, and write the result in the quotient.*

Give analysis Rule for the division of polynomials.

III. *Multiply the whole divisor by the quotient thus found, and subtract the product from the dividend.*

IV. *Arrange the remainder for a new dividend, with which proceed as before, till the first term of the divisor is no longer contained in the first term of the remainder.*

V. *Write the final remainder, if there be any, over the divisor in the form of a fraction, and the entire result will be the quotient sought.*

EXAMPLES FOR PRACTICE.

2. Divide $a^2 + 2ax + x^2$ by $a + x$. *Ans.* $a + x$.
3. Divide $a^3 - 3a^2y + 3ay^2 - y^3$ by $a - y$.
Ans. $a^2 - 2ay + y^2$.
4. Divide $a^3 + 5a^2b + 5ab^2 + b^3$ by $a + b$.
Ans. $a^2 + 4ab + b^2$.
5. Divide $x^3 - 3x^2z + z^3$ by $x - z$.
Ans. $x^2 - 2xz - 2z^2 - \frac{z^3}{x - z}$.
6. Divide $a^3 + 2a^2b + 2ab^2 + b^3$ by $a^2 + ab + b^2$.
Ans. $a + b$.
7. Divide $x^3 - 9x^2 + 27x - 27$ by $x - 3$.
Ans. $x^2 - 6x + 9$.
8. Divide $6x^4 - 96$ by $6x - 12$.
Ans. $x^3 + 2x^2 + 4x + 8$.
9. Divide $6a^4 + 9a^2 - 15a$ by $3a^2 - 3a$.
Ans. $2a^2 + 2a + 5$.
10. Divide $25x^5 - x^3 - 2x^2 - 8x$ by $5x^2 - 4x$.
Ans. $5x^3 + 4x^2 + 3x + 2$.
11. Divide $18a^2 - 8b^2$ by $6a + 4b$. *Ans.* $3a - 2b$.
12. Divide $2x^3 - 19x^2 + 26x - 16$ by $x - 8$.
Ans. $2x^2 - 3x + 2$.
13. Divide $y^5 + 1$ by $y + 1$. *Ans.* $y^4 - y^3 + y^2 - y + 1$.
14. Divide $y^6 - 1$ by $y - 1$.
Ans. $y^5 + y^4 + y^3 + y^2 + y + 1$.
15. Divide $x^2 - a^2$ by $x - a$. *Ans.* $x + a$.

16. Divide $6a^3 - 3a^2b - 2a + b$ by $3a^2 - 1$.
Ans. $2a - b$.

17. Divide $y^6 - 3y^4x^2 + 3y^2x^4 - x^6$ by $y^3 - 3y^2x + 3yx^2 - x^3$.
Ans. $y^3 + 3y^2x + 3yx^2 + x^3$

18. Divide $64a^4b^6 - 25a^2b^8$ by $8a^2b^3 + 5ab^4$.
Ans. $8a^2b^3 - 5ab^4$.

19. Divide $2a^4 - 2x^4$ by $a - x$.
Ans. $2a^3 + 2a^2x + 2ax^2 + 2x^3$.

20. Divide $(a - x)^5$ by $(a - x)^2$. *Ans.* $(a - x)^3$

21. Divide $a^3 - 3a^2x + 3ax^2 - x^3$ *by* $a - x$.
Ans. $a^2 - 2ax + x^2$.

22. Divide $a^5 + 1$ by $a + 1$.
Ans. $a^4 - a^3 + a^2 - a + 1$.

23. Divide $b^6 - 1$ by $b - 1$.
Ans. $b^5 + b^4 + b^3 + b^2 + b + 1$.

24. Divide $48a^3 - 92a^2x - 40ax^2 + 100x^3$ by $3a - 5x$.
Ans. $16a^2 - 4ax - 20x^2$.

25. Divide $4d^4 - 9d^2 + 6d - 1$ by $2d^2 + 3d - 1$.
Ans. $2d^2 - 3d + 1$.

26. Divide $6a^4 + 4a^3x - 9a^2x^2 - 3ax^3 + 2x^4$ by $2a^2 + 2ax - x^2$.
Ans. $3a^2 - ax - 2x^2$.

27. Divide $3a^4 - 8a^2b^2 + 3a^2c^2 + 5b^4 - 3b^2c^2$ by $a^2 - b^2$.
Ans. $3a^2 - 5b^2 + 3c^2$.

28. Divide $2x^2 + 7xy + 6y^2$ by $x + 2y$.
Ans. $2x + 3y$.

29. Divide $2mx + 3nx + 10mn + 15n^2$ by $x + 5n$.
Ans. $2m + 3n$.

30. Divide $d^4 - 3d^2c - 10c^2$ by $d^2 - 5c$. *Ans.* $d^2 + 2c$.

31. Divide $m^2 - c^2 + 2cz - z^2$ by $m + c - z$.
Ans. $m - c + z$.

32. Divide $y^5 + 32z^5$ by $y + 2z$.
Ans. $y^4 - 2y^3z + 4y^2z^2 - 8yz^3 + 16z^4$.

33. Divide $12(a + b)^3 + 3(a + b)^2$ by $3(a + b)$.
Ans. $4(a + b)^2 + a + b$.

34. Divide $3c(m - 5c)^2 - (m - 5c)^3$ by $(m - 5c)^2$.
Ans. $8c - m$.

GENERAL PRINCIPLES OF DIVISION.

69. The value of a quotient in division depends upon the relative values of the dividend and divisor; and the sign of the quotient depends upon the relative signs of the dividend and divisor. Hence any change in the value or the sign of either dividend or divisor must produce a change in the value or the sign of the quotient; though certain changes may be made in both dividend and divisor, at the same time, that will not affect the quotient. The laws that govern these changes are called *General Principles of Division.*

CHANGE OF VALUE.

70. It will be necessary to examine only those changes of value produced by multiplying and dividing the dividend and divisor.

Let us take $abcd$ for a dividend, and ab for a divisor; the quotient will be cd, and the operations performed upon dividend and divisor will affect this quotient as follows:

	Dividend.		Divisor.		Quotient.	
	$abcd$	$\div$	ab	$=$	cd	
1.	$abcde$	$\div$	ab	$=$	cde	Multiplying the dividend by e multiplies the quotient by e.
2.	abc	$\div$	ab	$=$	c	Dividing the dividend by d divides the quotient by d.
3.	$abcd$	$\div$	abc	$=$	d	Multiplying the divisor by c divides the quotient by c.
4.	$abcd$	$\div$	a	$=$	bcd	Dividing the divisor by b multiplies the quotient by b.
5.	$abcde$	$\div$	abe	$=$	cd	Multiplying both terms by e does not alter the quotient.
6.	bcd	$\div$	b	$=$	cd	Dividing both terms by a does not alter the quotient.

What determines the value of a quotient in division?

In these six operations, the factors employed to operate with are literal quantities, and may represent any numbers whatever; hence the results are general truths; they may be stated as follows:

PRIN. I. *Multiplying the dividend multiplies the quotient, and dividing the dividend divides the quotient.* (1 and 2.)

PRIN. II. *Multiplying the divisor divides the quotient, and dividing the divisor multiplies the quotient.* (3 and 4.)

PRIN. III. *Multiplying or dividing both dividend and divisor by the same quantity does not alter the quotient* (5 and 6.)

71. These three principles may be embraced in one

GENERAL LAW.

A change in the dividend *produces a* LIKE *change in the quotient; but a change in the* divisor *produces an* OPPOSITE *change in the quotient.*

CHANGE OF SIGN.

72. To investigate the relative changes of signs in division, let it be remembered that when the divisor and dividend have *like signs,* the quotient is *plus,* and when they have *unlike signs,* the quotient is *minus.* Then

1*st.* Suppose the divisor and dividend have *like* signs; if *either* of the signs be changed, they will become *unlike,* and the sign of the quotient will be changed from *plus* to *minus.*

2*d.* Suppose the divisor and dividend have *unlike* signs; if *either* of the signs be changed, they will become *alike,* and the sign of the quotient will be changed from *minus* to *plus.*

3*d.* Suppose again that the divisor and dividend have *like* signs; if *both* signs be changed at once, they will still be *alike,* and the sign of the quotient will remain *plus.*

Explain the principles which govern changes of value of the quotient in division. Repeat Principle I. Prin. II. Prin. III. The general law. Explain the principles which govern changes of signs, of divisor and quotient.

4th. Suppose again that the divisor and dividend have *unlike* signs; if *both* signs be changed at once, they will still be unlike, and the sign of the quotient will remain *minus.*

These results may be embraced in two principles, as follows:

PRIN. I. *Changing the sign of either dividend or divisor, changes the sign of the quotient.*

PRIN. II. *Changing the signs of both dividend and divisor, does not alter the sign of the quotient.*

NOTE. — If the dividend or divisor is a polynomial, its entire value is changed by changing the signs of all its terms.

EXACT DIVISION.

73. An *Exact Division* is one in which the quotient has no fractional part.

74. From the rule for division (**66**) it is evident that the exact division of one monomial by another will be impossible: —

1*st. When the coefficient of the divisor is not exactly contained in the coefficient of the dividend.*

2*d. When a literal factor has a greater exponent in the divisor than in the dividend.*

3*d. When a literal factor of the divisor is not found in the dividend.*

75. It is also evident (**68**) that the division of one polynomial by another will be impossible:

1*st. When the first term of the divisor arranged with reference to any one of its letters is not exactly contained in the first term of the dividend arranged with reference to the same letter.*

Repeat Prin. I. Prin. II. Define an exact divisor. Explain under what circumstances and why the exact division of one monomial by another is impossible. Under what circumstances and why is one polynomial not an exact divisor of another?

2d When a remainder occurs, having no term which will exactly contain the first term of the divisor.

76. In all cases where exact division is impossible, the quotient may be indicated by writing the dividend above a horizontal line, and the divisor below, according to definition (**7**).

RECIPROCALS, ZERO POWERS, AND NEGATIVE EXPONENTS.

77. The **Reciprocal** of a quantity is 1 divided by that quantity; thus $\frac{1}{a}$ is the reciprocal of a; $\frac{1}{x-y}$ is the reciprocal of $x-y$.

78. If, in the division of powers, we conform strictly to the rule of subtracting the exponent of the divisor from the exponent of the dividend, then, in the case of equal powers, the exponent of the quotient will be 0, and in cases where the divisor is the higher power, the exponent of the quotient will become negative.

79. To explain the import of a cipher when used as an exponent, we observe that the quotient of any quantity divided by itself is 1; consequently, when the divisor and dividend are like powers of the same quantity, we may have two expressions for the quotient; thus

$$\frac{a}{a} = a^{1-1} = a^0, \text{ or } \frac{a}{a} = 1;$$

$$\frac{a^m}{a^m} = a^{m-m} = a^0, \text{ or } \frac{a^m}{a^m} = 1.$$

Therefore, (Ax. 7), $\qquad a^0 = 1.$

In the cases previously stated how may the quotient be written? Define the reciprocal of a quantity. In Division when will the exponent of the quotient be 0? When negative? Explain why. What is the value of any quantity whose exponent is 0? Why?

But a may represent any quantity whatever. Hence,

Any quantity having a cipher for an exponent is equal to unity.

NOTE.—When a quantity with a cipher for an exponent is a *factor* in an algebraic expression, it may be suppressed without affecting the value of the expression; yet it is frequently retained in order to indicate the process by which the result was obtained.

80. To show the signification of negative exponents, let us divide a^5 by a^7, by taking the difference of the exponents; thus, $a^5 \div a^7 = a^{5-7} = a^{-2}$.

But the value of the quotient will not be altered if we divide both dividend and divisor by a^5 (**70**, III); thus,

$$a^5 \div a^7 = 1 \div a^2 = \frac{1}{a^2}.$$

These quotients being equal, we have

$$a^{-2} = \frac{1}{a^2}.$$

This principle may also be illustrated as follows. Since the zero power of any factor is 1, we may have

$$\frac{a^0}{a^2} = a^{0-2} = a^{-2}, \text{ or } \frac{a^0}{a^2} = \frac{1}{a^2}; \text{ hence, } a^{-2} = \frac{1}{a^2};$$

$$\frac{a^0}{a^m} = a^{0-m} = a^{-m}, \text{ or } \frac{a^0}{a^m} = \frac{1}{a^m}; \text{ hence, } a^{-m} = \frac{1}{a^m}.$$

From these illustrations we deduce the following inference:

Any quantity having a negative exponent is equal to the reciprocal of that quantity with an equal positive exponent.

What do negative exponents signify? What relation do they bear to reciprocals.

FACTORING.

81. The **Factors** of a quantity are those quantities which, being multiplied together, will produce the given quantity.

82. A **Composite Quantity** is one that may be produced by the multiplication of two or more factors. A composite quantity is exactly divisible by any of its factors.

83. A **Prime Quantity** is one that cannot be produced by the multiplication of two or more factors, and is divisible only by itself and unity.

Several quantities are *prime to each other* when they have no common factor, or when no quantity except unity will divide them all.

CASE I.

84. To factor a monomial.

The prime factors of a purely algebraic quantity, consisting of a single term, *are visible to the eye;* and this is one of the principal advantages of an algebraic expression. Algebraic quantities are factored by inspection or by trial, the same as numbers in arithmetic.

1. What are the prime factors of $6a^3b^2c$?

OPERATION.

$$\begin{array}{l} 6 = 2 \times 3 \\ a^3 = a \times a \times a \\ b^2 = b \times b \\ c = c \\ \hline 2 \times 3aaabbc \end{array}$$

ANALYSIS. The prime factors of 6 are 2 and 3; the exponents show that a is taken 3 times as a factor in the given term, b twice, and c once; and $6a^3b^2c = 2 \times 3aaabbc$. Hence,

RULE. *Resolve the numeral coefficient into its prime factors, and write each letter as many times as there are units in its exponent.*

Define factors. A composite quantity. A prime quantity. Quantities prime to each other What is Case I? Give analysis. Rule.

EXAMPLES FOR PRACTICE.

2. Resolve $10x^2y^3$ into its prime factors.

Ans. $2 \times 5xxyyy$.

3. Resolve $15m^3c^4$ into its prime factors.

4. Resolve $24p^4z^2$ into its prime factors.

Ans. $2 \times 2 \times 2 \times 3ppppzz$.

5. Resolve $75a^2b^2cd^3$ into its prime factors.

6. Resolve $26m^4x^2yz$ into its prime factors.

Ans. $2 \times 13mmmmxxyz$.

CASE II.

85. To resolve a polynomial into two factors, a monomial and a polynomial.

Polynomials may be factored by inspection under certain conditions. If the terms have a common factor, the quantity may be separated into two factors, a monomial and a polynomial.

1. Resolve $2ax - 2am + 6az$ into its factors.

ANALYSIS. Since $2a$ is a factor common to all the terms, we divide by this factor, and obtain for a quotient, $x - m + 3z$, which is the other factor of the given quantity; or, $2ax - 2am + 6az = 2a(x - m + 3z)$. Hence,

RULE. *Divide by the greatest factor common to all the terms, inclose the quotient in a parenthesis, and write the divisor as the coefficient.*

EXAMPLES FOR PRACTICE.

2. Find the factors of $ax + bx$. *Ans.* $x(a + b)$.

3. Find the factors of $x + ax$. *Ans.* $x(1 + a)$.

4. Resolve $am + an + ax$ into its factors.

Ans. $a(m + n + x)$.

5. Resolve $bc^2 - bcx - bcy$ into its factors.

Ans. $bc(c - x - y)$.

What is Case II? Give analysis. Rule.

6. Resolve $4x^2 - 6xy$ into its factors.

Ans. $2x(2x - 3y)$.

7. Factor $a^2b^3 - a^2c - 2a^2d$. *Ans.* $a^2(b^3 - c - 2d)$.

8. Factor $3m^2z - 4my + 2c^2m$.

Ans. $m(3mz - 4y + 2c^2)$.

9. Factor $12c^4bx^3 - 15c^3x^4 - 6c^2x^3y$.

Ans. $3c^2x^3(4c^2b - 5cx - 2y)$.

10. Factor $cx - 3cxz + cx^2$. *Ans.* $cx(1 - 3z + x)$.

NOTE. — It may happen that a portion of a polynomial can be factored when there is no factor common to all the terms.

11. Factor $x^2 + 2bx - 6bc$. *Ans.* $x^2 + 2b(x - 3c)$.

12. Factor $a^2n + ma + mb$. *Ans.* $a^2n + m(a + b)$.

13. Factor $ax^2 + 3a^2x + bx^2 + 3b^2x$.

$$Ans. \begin{cases} ax(x + 3a) + bx(x + 3b); \text{ or,} \\ x^2(a + b) + 3x(a^2 + b^2). \end{cases}$$

14. Factor $a^3 + a^2b + ab^2 + b^3$.

$$Ans. \begin{cases} a(a^2 + ab + b^2) + b^3; \text{ or,} \\ a^3 + b(a^2 + ab + b^2). \end{cases}$$

15. Factor $x^3z^2 + x^2z^3 + x^2z + xz^2$.

$$Ans. \begin{cases} x^2z^2(x + z) + xz(x + z); \text{ or,} \\ (x^2z^2 + xz)(x + z); \text{ or,} \\ xz(xz + 1)(x + z). \end{cases}$$

16. Factor $ax + ay + bx + by$.

$$Ans. \begin{cases} a(x + y) + b(x + y); \text{ or,} \\ (a + b)(x + y). \end{cases}$$

CASE III.

86. To resolve a trinomial into two equal binomial factors.

A trinomial may be resolved into two binomial factors when two of its terms are perfect squares and positive, and the other

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What is Case III?

term is twice the product of their square roots, and either positive or negative.

1. Factor $a^2 + 2ac + c^2$.

ANALYSIS. a^2 is the square of a, c^2 is the square of c, and $2ac$ is twice the product of a and c; and since $a^2 + 2ac + c^2$ is the sum of the squares of a and c plus twice their product, it must be the square of $a + c$ (60); or $a^2 + 2ac + c^2 = (a + c)(a + c)$. Hence,

RULE. ***Connect the square roots of the two squares by the sign of the other term, and write the result twice as a factor.***

EXAMPLES FOR PRACTICE.

2. Resolve $a^2 + 2ax + x^2$ into its factors.
   Ans. $(a + x)(a + x)$.
3. Resolve $a^2 - 2ax + x^2$ into its factors.
   Ans. $(a - x)(a - x)$.
4. Resolve $A^2 - 2AB + B^2$ into its factors.
   Ans. $(A - B)(A - B)$.
5. Resolve $P^2 + 2PQ + Q^2$ into its factors.
   Ans. $(P + Q)(P + Q)$.
6. Resolve $9a^2 + 12ab + 4b^2$ into its factors.
   Ans. $(3a + 2b)(3a + 2b)$.
7. Resolve $4m^2 - 4m + 1$ into its factors.
   Ans. $(2m - 1)(2m - 1)$.
8. Resolve $4c^2 - 4cd + d^2$ into its factors.
   Ans. $(2c - d)(2c - d)$.
9. Factor $9m^2 + 12m + 4$. *Ans.* $(3m + 2)(3m + 2)$.
10. Factor $1 - 12z + 36z^2$. *Ans.* $(1 - 6z)(1 - 6z)$.
11. Factor $a^2c^2 - 2ac + 1$. *Ans.* $(ac - 1)(ac - 1)$.
12. Factor $x^4 + 2ax^3 + a^2x^2$. *Ans.* $(x^2 + ax)(x^2 + ax)$.
13. Factor $y^6 - 2y^3z^3 + z^6$ *Ans.* $(y^3 - z^3)(y^3 - z^3)$.

Give analysis. Rule.

CASE IV.

87. To resolve a binomial into two binomial factors.

A binomial may be resolved into two binomial factors, when both of its terms are perfect squares, and have contrary signs.

1. Factor $a^2 - b^2$.

ANALYSIS. a^2 is the square of a, b^2 is the square of b; and since $a^2 - b^2$ is the difference of the squares of a and b, it must be equal to the product of the sum and difference of a and b (61); or $a^2 - b^2 = (a + b)(a - b)$. Hence,

RULE. *Write the sum and difference of the square roots of the two given terms, as two binomial factors.*

EXAMPLES FOR PRACTICE.

2. Factor $x^2 - y^2$. *Ans.* $(x + y)(x - y)$.
3. Factor $m^2 - n^2$. *Ans.* $(m + n)(m - n)$.
4. Factor $y^2 - 4z^2$ *Ans.* $(y + 2z)(y - 2z)$.
5. Factor $4a^2 - 9b^2$. *Ans.* $(2a + 3b)(2a - 3b)$.
6. Factor $25c^2 - 1$. *Ans.* $(5c + 1)(5c - 1)$.
7. Factor $36c^4d^2 - 16m^6$. *Ans.* $(6c^2d + 4m^3)(6c^2d - 4m^3)$.
8. Factor $9a^2c^4x^2 - 1$. *Ans.* $(3ac^2x + 1)(3ac^2x - 1)$.
9. Factor $a^2z^2 - a^2y^2$. *Ans.* $(az + ay)(az - ay)$.
10. Factor $a^4 - c^4$. *Ans.* $(a^2 + c^2)(a + c)(a - c)$.
11. Factor $x^4 - y^4$. *Ans.* $(x^2 + y^2)(x + y)(x - y)$.
12. Factor $x^8 - z^8$. *Ans.* $(x^4 + z^4)(x^2 + z^2)(x + z)(x - z)$.
13. Factor $m^{16} - c^{16}$. *Ans.* $(m^8+c^8)(m^4+c^4)(m^2+c^2)(m+c)(m-c)$.
14. Factor $c^{32} - 1$. *Ans.* $(c^{16}+1)(c^8+1)(c^4+1)(c^2+1)(c+1)(c-1)$
15. Factor $a^2c - c$. *Ans.* $c(a + 1)(a - 1)$.
16. Factor $a^2c^2 - c^2$. *Ans.* $c^2(a + 1)(a - 1)$.
17. Factor $x^2y^2z^2 - x^2y^2$. *Ans.* $x^2y^2(z + 1)(z - 1)$.

What is Case IV? Give analysis. Rule.

18. Factor $x^9 - x$.

Ans. $x(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)$.

19. Factor $m^9 - m^5$.

Ans. $m^5(m^2 + 1)(m + 1)(m - 1)$.

GREATEST COMMON DIVISOR.

88. A **Common Divisor** of two or more quantities is a quantity that will exactly divide each of them.

89. The **Greatest Common Divisor** of two or more quantities is the greatest quantity that will exactly divide each of them.

90. It is evident that if two or more quantities be divided by their greatest common divisor, the quotients will be *prime to each other.*

1. What is the greatest common divisor of $4a^2b^3cd$, $48a^4b^2c^3x$, and $12a^3b^2cd^2$?

OPERATION.

$$\begin{array}{rcccccccccccc}
4a^2b^3cd & = & & & 2^2 & \times & a^2 & \times & b^3 & \times & c & \times & d & & \\
48a^4b^2c^3x & = & 3 & \times & 2^4 & \times & a^4 & \times & b^2 & \times & c^3 & & & \times & x \\
12a^3b^2cd^2 & = & 3 & \times & 2^2 & \times & a^3 & \times & b^2 & \times & c & \times & d^2 & & \\
\hline
4a^2b^2c & = & & & 2^2 & \times & a^2 & \times & b^2 & \times & c & & & &
\end{array}$$

ANALYSIS. We resolve the quantities into their component factors, and write all the powers of the same factor under each other. By inspection we perceive that all the quantities contain at least the second power of 2, and we write 2^2 underneath as a factor of the greatest common divisor sought; all the quantities contain at least the second power of a, and we write a^2 underneath; all the quantities contain at least the second power of b and the first power of c, and we write these factors underneath; and, since these are all the common factors, their product, $4a^2b^2c$, must be the greatest common divisor of the given quantities.

2. What is the greatest common divisor of $3ac^2(x^4 - c^4)$, and $a^2cx^2 - a^2c^3$?

Define a common divisor. The greatest common divisor. Give analysis of Example 1.

OPERATION.

$$
\begin{array}{rl}
3ac^2(x^4 - c^4) = & 3 \times a \times c^2 \times (x^2 + c^2) \times (x + c) \times (x - c) \\
a^2cx^2 - a^2c^3 = & \quad a^2 \times c \times \qquad\qquad\qquad (x + c) \times (x - c) \\
\hline
ac(x^2 - c^2) = & \quad a \times c \times \qquad\qquad\qquad (x + c) \times (x - c)
\end{array}
$$

ANALYSIS. Resolving the quantities into factors as before, we readily perceive that the only common factors are a, c, $(x + c)$, and $(x - c)$; and the product of these factors, $ac(x^2 - c^2)$, must therefore be the greatest common divisor sought.

From these examples we deduce the following

RULE. I. *Resolve the given quantities into factors, and write all the powers of the same factor under each other.*

II. *Multiply together the lowest power of each common factor, and the product will be the greatest common divisor sought.*

EXAMPLES FOR PRACTICE.

3. What is the greatest common divisor of $4a^2c^2$, and $10abc^3$? *Ans.* $2ac^2$.

4. What is the greatest common divisor of $3abx^3$, and $12abx^4z$? *Ans.* $3abx^3$.

5. What is the greatest common divisor of $4a^3b^2x^5z^3$, and $8a^5x^2z^2$? *Ans.* $4a^3x^2z^2$.

6. What is the greatest common divisor of $4am^2y^4z^5$, $12m^5z^6$, and $16a^3m^3z^2$? *Ans.* $4m^2z^2$.

7. What is the greatest common divisor of $6a^2c^2d^2$, $12a^3c^4d^5$, $9a^5c^3d^4$, and $24a^3c^2dm$? *Ans.* $3a^2c^2d$.

8. Find the greatest common divisor of $a^2 - b^2$, and $a^2 - 2ab + b^2$. *Ans.* $a - b$.

9. Find the greatest common divisor of $a^2 - c^2$, and $a^2 + 2ac + c^2$. *Ans.* $a + c$.

10 Find the greatest common divisor of $m^2 - 2m$, and $2mn^2 - 4n^2$. *Ans.* $m - 2$.

Of Example 2. Rule.

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11. Find the greatest common divisor of $ax^2 - ay^2$, $am^2x - am^2y$, and $a^2x^2 - 2a^2xy + a^2y^2$. *Ans.* $a(x-y)$.

12. Find the greatest common divisor of $16a^2 - c^2$, and $16a^2 - 8ac + c^2$. *Ans.* $4a - c$.

13. Find the greatest common divisor of $3a^2b - 9a^2c - 18a^2mz$, and $b^2c - 3bc^2 - 6bcmz$. *Ans.* $b - 3c - 6mz$.

LEAST COMMON MULTIPLE.

91. A **Multiple** of a quantity is another quantity exactly divisible by it.

NOTE.—If a quantity be *multiplied* by any factor, the result is properly called a *multiple* of that quantity, and this is the real signification of the word multiple. But the product is always *divisible* by the multiplicand; hence the definition as given above.

92. A **Common Multiple** of two or more quantities is one which is exactly divisible by each of them.

93. The **Least Common Multiple** of two or more quantities is the least quantity exactly divisible by each of them.

94. It is evident that one quantity, to be divisible by several other quantities, must contain all the factors in each of the given quantities.

1. What is the least common multiple of $8a^2x^2y$, and $12a^3b^3x$?

OPERATION.

$$\begin{array}{rl} 8a^2x^2y &= \phantom{3 \times{}} 2^3 \times a^2 \times x^2 \times y \\ 12a^3b^3x &= 3 \times 2^2 \times a^3 \times x \phantom{{}\times y} \times b^3 \\ \hline 24a^3b^3x^2y &= 3 \times 2^3 \times a^3 \times x^2 \times y \times b^3 \end{array}$$

ANALYSIS. We resolve the given quantities into their component factors, and write the powers of each separate letter or factor under each other. The different prime factors are 3, 2, a, x, y, and b; and the least common multiple must contain not only each of these, but the *highest power* of each that is contained in the given quantities,

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Define a Multiple. A common multiple. The least common multiple. When one quantity is a common multiple of several others, what must it contain? Give analysis of Example 1.

otherwise it will not contain all the factors of the given quantities. By inspecting the powers of each, the highest are the 1st power of 3, 3d power of 2, 3d power of a, 2d power of x, 1st power of y, and 3d power of b; and their product is $24a^3b^3x^2y$, the least common multiple required. Hence, the

Rule. I. *Resolve the given quantities into factors, and write all the powers of the same factor under each other.*

II. *Multiply together the highest power of every factor, and the product will be the least common multiple sought.*

EXAMPLES FOR PRACTICE.

2. Find the least common multiple of $3ab^3c$, $5ab^2c$, abd^2, and $15a^2b^3c$. *Ans.* $15a^2b^3cd^2$.

3. Find the least common multiple of $6xy$, $9x^4z$, $3x^2y^2z$, and x^4z. *Ans.* $18x^4y^2z$.

4. Find the least common multiple of $2mn$, $3m^2z^3$, $6mz^4$, and $4mnz$. *Ans.* $12m^2nz^4$.

5. Required the least common multiple of $27a$, $15b$, $9ab$, and $3a^2$. *Ans.* $135a^2b$.

6. Find the least common multiple of $(a^2 - x^2)$, $4(a - x)$, and $(a + x)$. *Ans.* $4(a^2 - x^2)$.

7. Required the least common multiple of $a^2(a - x)$, and $ax^4(a^2 - x^2)$. *Ans.* $a^2x^4\ (a^2 - x^2)$.

8. Required the least common multiple of $x^2(x - y)$, a^4x^2, and $12axy^2$. *Ans.* $12a^4x^2y^2(x - y)$.

9. Required the least common multiple of $10a^2x^2(a - b)$, $15x^5(a + b)$, and $12(a^2 - b^2)$. *Ans.* $60a^2x^5(a^2 - b^2)$.

10. What is the least common multiple of $m^4 - 1$, $m^2 - 2m + 1$, and $m^2 + 2m + 1$? *Ans.* $m^6 - m^4 - m^2 + 1$.

11. What is the least common multiple of $x^2 - y^2$, $x^2y - xy^2$, and $x^2y + xy^2$? *Ans.* $x^3y - xy^3$.

12. What is the least common multiple of $m^2 - 4$, $zm - 2z$, and $m^2 + 2m$? *Ans.* $zm^3 - 4zm$.

Give Rule.

FRACTIONS.

95. The word *Fraction* relates to a certain mode or form of indicating division; and fractional forms have precisely the same signification in Algebra as in Arithmetic.

96. A **Fraction** is a quotient expressed by writing the dividend above a horizontal line, and the divisor below; thus, $\frac{a}{b}$ is a fraction, and is read, *a divided by b.*

97. The **Denominator** of a fraction is the quantity below the line, or the divisor.

98. The **Numerator** of a fraction is the quantity above the line, or the dividend.

99. Since a quantity is divided by dividing any one of its factors, we have $\frac{a}{b} = \frac{1 \times a}{b} = \frac{1}{b} \times a$; hence, *a fraction is equal to the reciprocal of its denominator multiplied by its numerator.*

100. An **Entire Quantity** is an algebraic expression which has no fractional part; as, $3a$, or $x - 3y^2$.

101. A **Mixed Quantity** is one which has both entire and fractional parts; as, $a + \frac{c}{a}$, $m - \frac{3c}{1-b}$.

SIGNS.

102. Each term in the numerator and denominator of a fraction has its own particular sign, distinct from the real sign of the fraction. Thus, in the fraction, $\frac{x^2 - 2xy + y^2}{x^2y - xy^2}$, the signs

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Define a fraction. A denominator. A numerator. A fraction is equal to the reciprocal of what? Define an entire quantity. A mixed quantity.

of a part of the terms only are expressed. If no sign is prefixed to the first term of a numerator or denominator, the plus sign is understood.

103. The **Apparent Sign** of a fraction is the sign written before the dividing line, to indicate whether the fraction is to be added or subtracted; thus, in $m + \frac{x^2 - ax}{a - x}$, the apparent sign of the fraction is plus, and indicates that the fraction is to be added to m.

104. The **Real Sign** of a fraction is the sign of its numerical value, when reduced to a monomial, and shows whether the fraction is essentially a positive or a negative quantity; thus, in the last fraction, $\frac{x^2 - ax}{a - x}$, let $x = 2$, and $a = 12$; then $\frac{x^2 - ax}{a - x} = \frac{4 - 12 \times 2}{12 - 2} = \frac{-20}{10} = -2$. Hence, the *real* sign of this fraction is *minus*, though its *apparent* sign is *plus*.

GENERAL PRINCIPLES OF FRACTIONS.

105. Since fractions indicate division, all changes in the numerator and denominator of a fraction will affect the value and sign of that fraction according to the laws of division; and we have only to modify the language of the General Principles of Division (**70**), by substituting the words *numerator*, *denominator*, and *fraction*, for the words *dividend*, *divisor*, and *quotient*, and we shall express the laws governing the changes in the value and sign of a fraction.

CHANGE OF VALUE.

PRIN. I. *Multiplying the numerator multiplies the fraction, and dividing the numerator divides the fraction.*

Define the apparent sign of a fraction. The real sign. Fractions always indicate what? Adapt the general principles of Division to fractions. Repeat the principles that govern change of value.

PRIN. II. *Multiplying the denominator divides the fraction, and dividing the denominator multiplies the fraction.*

PRIN. III. *Multiplying or dividing both numerator and denominator by the same quantity does not alter the value of the fraction.*

106. These three principles may be embodied in one

GENERAL LAW.

A change in the numerator *produces a* LIKE *change in the value of the fraction; but a change in the* denominator *produces an* OPPOSITE *change in the value of the fraction.*

CHANGE OF SIGN.

107. PRIN. I. *Changing the sign of either numerator or denominator, changes the real sign of the fraction.*

PRIN. II. *Changing the signs of both numerator and denominator at the same time, does not alter the real sign of the fraction.*

PRIN. III. *Changing the apparent sign of the fraction changes the real sign.*

REDUCTION.

108. The **Reduction** of a quantity is the operation of changing its form without altering its value.

CASE I.

109. To reduce a fraction to its lowest terms.

A fraction is in its *lowest terms* when its numerator and denominator are prime to each other.

1. Reduce $\frac{14ab^3c}{21a^2bc^2}$ to its lowest terms.

Apply the general law. The principles that govern change of signs. Define Reduction. What is Case I?

OPERATION.

$$\frac{14ab^3c}{21a^2bc^2} = \frac{2b^2}{3ac}$$

ANALYSIS. If we divide both numerator and denominator of this fraction by the same number, its value will not be changed (105, III); and if we divide by the greatest common divisor, the quotients will be prime to each other (90), and consequently the fraction will be in its lowest terms. By inspection we find $7abc$ to be the greatest common divisor; and dividing both terms by this quantity, we have $\frac{2b^2}{3ac}$, the answer.

2. Reduce $\frac{a^2x + ax^2}{a^2 - x^2}$ to its lowest terms.

OPERATION.

$$\frac{a^2x + ax^2}{a^2 - x^2} = \frac{ax\,(a + x)}{(a - x)\,(a + x)} = \frac{ax}{a - x}.$$

ANALYSIS. We first resolve the numerator and denominator into their prime factors, and then cancel the common factor $(a + x)$, and we have $\frac{ax}{a - x}$, the answer.

From these examples we deduce the following

RULE. *Divide both numerator and denominator by their greatest common divisor.* Or,

Resolve the numerator and denominator into their prime factors, and cancel all those that are common.

EXAMPLES FOR PRACTICE.

3. Reduce $\frac{12ax}{18ab}$ to its lowest terms. *Ans.* $\frac{2x}{3b}$.

4. Reduce $\frac{14a^2x^2y}{21ax^2}$ to its lowest terms. *Ans.* $\frac{2ay}{3}$.

Give analyses. Rule.

5. Reduce $\frac{116a^5x^2y}{68a^3xy^3}$ to its lowest terms. *Ans.* $\frac{29a^2x}{17y}$.

6. Reduce $\frac{51a^3b-63a^2b^2}{36a^4b^4-9ab}$ to its lowest terms.

Ans. $\frac{17a^2-21ab}{12a^3b^3-3}$.

7. Reduce $\frac{4a^2-4x^2}{3(a+x)}$ to its lowest terms.

Ans. $\frac{4(a-x)}{3}$.

8. Reduce $\frac{x^5-b^2x^3}{x^4-b^4}$ to its lowest terms. *Ans.* $\frac{x^3}{x^2+b^2}$.

9. Reduce $\frac{x^2-1}{xy+y}$ to its lowest terms. *Ans.* $\frac{x-1}{y}$.

10. Reduce $\frac{cx+cx^2}{acx+abx}$ to its lowest terms.

Ans. $\frac{c+cx}{ac+ab}$.

11. Divide $x^3y^2+x^2y^3$ by ax^2y+axy^2. *Ans.* $\frac{xy}{a}$.

12. Divide $4a+4b$ by $2a^2-2b^2$. *Ans.* $\frac{2}{a-b}$.

13. Divide n^3-2n^2 by n^2-4n+4. *Ans.* $\frac{n^2}{n-2}$.

14. Reduce $\frac{5a^2+5ax}{a^2-x^2}$ to its lowest terms.

15. Reduce $\frac{x^3-c^2x}{x^2+2cx+c^2}$ to its lowest terms.

16. Reduce $\frac{(x^2-a^2)x}{x^3-a^3}$ to its lowest terms.

17. Reduce $\frac{a^2x^4-a^2y^4}{x^4+x^2y^2}$ to its lowest terms.

CASE II.

110. To reduce a fraction to an entire or mixed quantity.

1. Reduce $\frac{ab + x}{b}$ to a mixed quantity.

OPERATION.

$$\frac{ab + x}{b} = a + \frac{x}{b}.$$

ANALYSIS. Since the value of the fraction is the quotient of the numerator divided by the denominator, we perform the division indicated, and obtain a for the entire part of the quotient, and $+\frac{x}{b}$ for the fractional part. Hence, the

RULE. I. *Divide the numerator by the denominator as far as possible, for the entire part.*

II. *Write the remainder over the denominator, and annex the fraction thus found to the entire part, with its proper sign.*

NOTE. — If any term be found in the numerator, whose *literal part* is exactly divisible by some term in the denominator, and having a greater coefficient than this term of the denominator, the reduction will be possible; otherwise, it will be impossible.

EXAMPLES FOR PRACTICE.

2. Reduce $\frac{19}{8}$ and $\frac{a^2 + bx}{a}$ to mixed quantities.

Ans. $2\frac{3}{8}$ and $a + \frac{bx}{a}$.

3. Reduce $\frac{5ay + ab + x}{y}$ to a mixed quantity.

Ans. $5a + \frac{ab + x}{y}$.

4. Reduce $\frac{2a^2 - 2b^2}{a - b}$ to an entire quantity.

Ans. $2a + 2b$.

Give Case II. Analysis. Rule.

5. Reduce $\frac{15a^3 - 2x}{5a^2}$ to a mixed quantity.

Ans. $3a - \frac{2x}{5a^2}$.

6. Reduce $\frac{a^2 + ab + b^2}{a}$ to a mixed quantity.

Ans. $a + b + \frac{b^2}{a}$.

7. Reduce $\frac{12a^2 + 4a - 3c}{4a}$ to a mixed quantity.

Ans. $3a + 1 - \frac{3c}{4a}$.

8. Reduce $\frac{10cx + a - b}{2x}$ to a mixed quantity.

Ans. $5c + \frac{a - b}{2x}$.

9. Reduce $\frac{x^2 + 2xy + y^2 + x}{x + y}$ to a mixed quantity.

Ans. $x + y + \frac{x}{x + y}$.

10. Reduce $\frac{x^2 - 6c^2d - m}{3cd}$ to a mixed quantity

Ans. $\frac{x^2 - m}{3cd} - 2c$.

11. Reduce $\frac{a^2 + 7ab^2}{3ab}$ to a mixed quantity.

Ans. $2b + \frac{a^2 + ab^2}{3ab}$.

CASE III.

111. To reduce any fraction to the form of an entire quantity.

It is evident that if an algebraic quantity be in the form of a fraction, and the fraction in its lowest terms, it will not reduce to an entire quantity by the last case. But the principle of negative exponents enables us to express the value of any fraction whatever in the form of an entire quantity.

1. Reduce $\frac{a}{c^2}$ to the form of an entire quantity.

Give Case III.

OPERATION.

$$\frac{a}{c^3} = a \times \frac{1}{c^3} = a \times c^{-3} = ac^{-3}$$

ANALYSIS. It has been shown that a fraction is equal to the product of its numerator into the reciprocal of its denominator; or $\frac{a}{c^3} = a \times \frac{1}{c^3}$ (99). But $\frac{1}{c^3} = c^{-3}$ (80); whence the expression becomes $a \times c^{-3} = ac^{-3}$.

From this example we deduce the following

RULE. *Reduce the fraction to its lowest terms, and then multiply its numerator by the reciprocal of its denominator expressed by negative exponents.*

EXAMPLES FOR PRACTICE.

Reduce the following fractions to the form of entire quantities: —

2. $\frac{a^2b}{c^3}$. *Ans.* a^2bc^{-3}.

3. $\frac{m^2}{ab^2c^3}$. *Ans.* $m^2a^{-1}b^{-2}c^{-3}$.

4. $\frac{3a^2}{2b^2c}$. *Ans.* $3 \times 2^{-1}a^2b^{-2}c^{-1}$.

5. $\frac{a^2x^3cm}{ax^5cm^4}$. *Ans.* $ax^{-2}m^{-3}$.

6. $\frac{x - y}{x + y}$. *Ans.* $(x - y)\ (x + y)^{-1}$.

7. $\frac{a^2 + 2ac + c^2}{a^2 - c^2}$. *Ans.* $(a + c)\ (a - c(^{-1}$.

8. $\frac{m^2z}{a^2m - 2am^2 + m^3}$. *Ans.* $mz\ (a - m)^{-2}$.

9. $\frac{x^4 - 2x^2z^2 + z^4}{x^6 - z^4x^2}$. *Ans.* $(x^2 - z^2)\ (x^2 + z^2)^{-1}x^{-2}$.

Give Analysis. Rule.

112. Since a factor with a positive exponent may be transferred from the denominator to the numerator by making its exponent negative, a factor with a negative exponent may be transferred by making its exponent positive. Hence we have this general conclusion:

A factor may be transferred from either term of a fraction to the other, by changing the sign of its exponent.

EXAMPLES FOR PRACTICE.

1. Reduce $\dfrac{a^2b}{c^{-2}}$ to the form of an entire quantity. *Ans.* a^2bc^2.

2. Reduce $\dfrac{3xy^2}{m^{-2}x^{-3}}$ to the form of an entire quantity. *Ans.* $3x^4y^2m^2$.

3. Reduce $\dfrac{c(a-m)}{(a+m)^{-1}}$ to the form of an entire quantity. *Ans.* $c(a^2-m^2)$.

4. Reduce $\dfrac{a^2bc^{-2}}{ab^{-1}c^{-5}}$ to the form of an entire quantity. *Ans.* ab^2c^3.

5. Reduce $\dfrac{x^{-2}b^3}{cz^{-2}}$ to positive exponents. *Ans.* $\dfrac{b^3z^2}{cx^2}$.

6. Reduce $\dfrac{m(x-y)^{-1}}{x+y}$ to positive exponents. *Ans.* $\dfrac{m}{x^2-y^2}$.

Reduce the following fractions to forms having only known quantities in the numerators and unknown quantities in the denominators.

7. $\dfrac{mx^2}{b^3y}$. *Ans.* $\dfrac{mb^{-3}}{x^{-2}y}$.

8. $\dfrac{c^2x^{-3}}{c^{-3}xz}$. *Ans.* $\dfrac{c^5}{x^4z}$.

9. $\dfrac{a^2bx^3}{b^3x^{-2}}$. *Ans.* $\dfrac{a^2b^{-2}}{x^{-5}}$.

How may a factor be transferred from one term of a fraction to the other?

CASE IV

113. To reduce a mixed quantity to a fraction.

1. Reduce $2\frac{3}{8}$ to a fraction.

ANALYSIS. $2=\frac{16}{8}$; and $2\frac{3}{8}=\frac{16}{8}+\frac{3}{8}=\frac{16+3}{8}=\frac{19}{8}$.

2. Reduce $a+\frac{x}{b}$ to a fraction.

OPERATION.

$$a+\frac{x}{b}=\frac{ab+x}{b}$$

ANALYSIS. It is evident that $a=\frac{ab}{b}$. But $\frac{ab}{b}=ab\times\frac{1}{b}$, (99); also $\frac{x}{b}=x\times\frac{1}{b}$; and ab times $\frac{1}{b}$ added to x times $\frac{1}{b}$ is equal to $ab+x$ times $\frac{1}{b}$, or $(ab+x)\times\frac{1}{b}$, which is equal to $\frac{ab+x}{b}$, the answer.

The algebraic operation is exactly like the arithmetical, and governed by the same principle. Hence, the

RULE. *Multiply the entire part by the denominator of the fraction; add the numerator if the sign of the fraction be plus, and subtract it if the sign be minus, and write the result over the denominator.*

EXAMPLES FOR PRACTICE.

3. Reduce $7\frac{1}{7}$ and $ax+\frac{b}{c}$ to fractions.

Ans. $\frac{50}{7}$ and $\frac{acx+b}{c}$.

4. Reduce $3-\frac{1}{2}$ and $x^2-\frac{x}{y}$ to fractions.

Ans. $\frac{5}{2}$ and $\frac{x^2y-x}{y}$.

5. Reduce $y-1+\frac{1-y}{1+y}$ to a fractional form.

Ans. $\frac{y^2-y}{y+1}$.

Give Case IV. Analysis. Rule.

6. Reduce $x + y + \dfrac{a}{x+y}$ to the form of a fraction.

Ans. $\dfrac{x^2 + 2xy + y^2 + a}{x+y}$.

7. Reduce $4 + 2x + \dfrac{b}{c}$ to a fraction.

Ans. $\dfrac{4c + 2cx + b}{c}$.

8. Reduce $5x - \dfrac{2x+5}{3}$ to a fraction. *Ans.* $\dfrac{13x - 5}{3}$.

9. Reduce $3a - 9 - \dfrac{3a^2 - 30}{a+3}$ to a fraction.

Ans. $\dfrac{3}{a+3}$.

10. Reduce $x + \dfrac{2ax + a^2}{x}$ to a fraction. *Ans.* $\dfrac{(x+a)^2}{x}$.

11. Reduce $a + b + \dfrac{c^2}{a+b}$ to a fraction.

Ans. $\dfrac{(a+b)^2 + c^2}{a+b}$.

12. Reduce $a + x + \dfrac{2x^2}{a-x}$ to a fraction. *Ans.* $\dfrac{a^2 + x^2}{a-x}$.

13. Reduce $a - \dfrac{ax}{a-x}$ to a fraction. *Ans.* $\dfrac{a^2 - 2ax}{a-x}$.

CASE V.

114. To reduce fractions to a common denominator.

1. Reduce $\dfrac{a}{x}$, $\dfrac{b}{y}$, and $\dfrac{c}{z}$, to a common denominator.

OPERATION.

$$\frac{a}{x} = \frac{ayz}{xyz}$$
$$\frac{b}{y} = \frac{xbz}{xyz}$$
$$\frac{c}{z} = \frac{xyc}{xyz}$$

ANALYSIS. We multiply both terms of each fraction by the denominators of the others; that is, the terms of the first by yz, the terms of the second by xz, and the terms of the third by xy. This process cannot alter the values of the fractions (**105**, III.,) and it must reduce them to a common denominator, because each new denominator is necessarily the product of all the given denominators. Hence, the

What is Case V? Give analysis.

RULE. *Multiply each numerator by all the denominators except its own, for the new numerators; and all the denominators together for a common denominator.*

NOTE.—Mixed quantities must first be reduced to fractions, and entire quantities to fractional forms by writing 1 for a denominator.

EXAMPLES FOR PRACTICE.

2. Reduce $\frac{3x}{2a}$, $\frac{2b}{3c}$, and d, to a common denominator.

Ans. $\frac{9cx}{6ac}$, $\frac{4ab}{6ac}$, and $\frac{6acd}{6ac}$.

3. Reduce $\frac{a}{m^2}$, $\frac{bc}{mx}$, and $\frac{m}{c}$, to a common denominator.

Ans. $\frac{acmx}{m^2xc}$, $\frac{bc^2m^2}{m^2xc}$, and $\frac{m^4x}{m^2xc}$.

4. Reduce $\frac{3}{4}$, $\frac{2x}{3}$, and $a+\frac{2x}{a}$, to a common denominator.

Ans. $\frac{9a}{12a}$, $\frac{8ax}{12a}$, and $\frac{12a^2+24x}{12a}$.

5. Reduce $\frac{a}{x-y}$, $\frac{m}{x+y}$, and $\frac{z}{x^2+y^2}$, to a common denominator.

Ans. $\frac{a(x^3+xy^2+x^2y+y^3)}{x^4-y^4}$, $\frac{m(x^3+xy^2-x^2y-y^3)}{x^4-y^4}$, $\frac{z(x^2-y^2)}{x^4-y^4}$.

6. Reduce $\frac{a+c}{x}$, $\frac{x}{a-c}$, and $\frac{a}{b}$, to a common denominator.

Ans. $\frac{b(a^2-c^2)}{bx(a-c)}$, $\frac{bx^2}{bx(a-c)}$, and $\frac{ax(a-c)}{bx(a-c)}$.

CASE VI.

115. To reduce fractions to their least common denominator.

Since a fraction can be reduced to higher terms only by multiplication, each of the higher denominators it may have

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Give Rule. Repeat Case VI.

must be some multiple of its lowest denominator. Hence, a *common* denominator for two or more fractions must be a *common multiple* of their lowest denominators, and the *least* common denominator must be the *least* common multiple.

1. Reduce $\frac{c}{ab^2}$ and $\frac{m}{a^2b}$ to their least common denominator.

OPERATION.

$$a^2b^2 \div ab^2 = a; \text{ and } c \times a = ac.$$
$$a^2b^2 \div a^2b = b; \text{ and } m \times b = bm.$$
$$\frac{c}{ab^2} = \frac{ac}{a^2b^2}; \text{ and } \frac{m}{a^2b} = \frac{bm}{a^2b^2}.$$

ANALYSIS. We find by inspection that a^2b^2 is the least common multiple of the given denominators; it is, therefore, the least denominator to which the fractions can be reduced. To ascertain what factor will reduce each denominator to a^2b^2, we divide this term by each denominator, and obtain a and b. Since the given denominators must be multiplied by a and b respectively to reduce them to the required denominator, the corresponding numerators must also be multiplied by these factors for the new numerators; and we have $c \times a$, or ac, for the first numerator, and $m \times b$, or bm, for the second numerator, and $\frac{ac}{a^2b^2}$ and $\frac{bm}{a^2b^2}$, the answer.

Hence the following

RULE. I. *Find the least common multiple of all the denominators for the least common denominator.*

II. *Divide this common denominator by each of the given denominators, and multiply each numerator by the corresponding quotient. The products will be the new numerators.*

EXAMPLES FOR PRACTICE.

Reduce the following fractions and mixed quantities to their least common denominator.

2. $\frac{m}{ac}$, $\frac{x}{b^2c}$, and $\frac{z}{c^2d}$. *Ans.* $\frac{b^2cdm}{ab^2c^2d}$, $\frac{acdx}{ab^2c^2d}$, and $\frac{ab^2z}{ab^2c^2d}$.

Give analysis. Rule.

3. $\frac{m}{a^2}$, $\frac{c+m}{ac}$, and $\frac{d}{ab}$. *Ans.* $\frac{mbc}{a^2bc}$, $\frac{abc+abm}{a^2bc}$, and $\frac{acd}{a^2bc}$.

4. $\frac{a+b}{3a^2}$, $\frac{a-b}{2ax^2}$, and $\frac{a^2}{4cx}$.

Ans. $\frac{4acx^2+4bcx^2}{12a^2cx^2}$, $\frac{6a^2c-6abc}{12a^2cx^2}$, and $\frac{3a^4x}{12a^2cx^2}$.

5. $\frac{c-d}{ab}$, $\frac{x}{a^2}$, and m. *Ans.* $\frac{ac-ad}{a^2b}$, $\frac{bx}{a^2b}$, and $\frac{a^2bm}{a^2b}$.

6. $a+\frac{x}{y}$, $\frac{c}{xy}$, and x. *Ans.* $\frac{axy+x^2}{xy}$, $\frac{c}{xy}$, and $\frac{x^2y}{xy}$.

7. $\frac{a}{x-y}$, $\frac{b}{x+y}$, and $\frac{c}{x^2-y^2}$.

Ans. $\frac{a(x+y)}{x^2-y^2}$, $\frac{b(x-y)}{x^2-y^2}$, and $\frac{c}{x^2-y^2}$.

8. $\frac{x}{x-1}$, $\frac{x^2}{x^2-1}$, and $\frac{x^4}{x^4-1}$.

Ans. $\frac{x^4+x^3+x^2+x}{x^4-1}$, $\frac{x^4+x^2}{x^4-1}$, and $\frac{x^4}{x^4-1}$.

9. $\frac{a-b}{ac}$ and $\frac{a-b}{a(a+b)}$.

Ans. $\frac{a^2-b^2}{ac(a+b)}$ and $\frac{c(a-b)}{ac(a+b)}$.

10. $\frac{a}{b(1-m^2)}$ and $\frac{a}{c(1-m)}$.

Ans. $\frac{ac}{bc(1-m^2)}$ and $\frac{ab(1+m)}{bc(1-m^2)}$.

ADDITION.

116. We have seen (**50**), that entire quantities may be added when they have a common factor, to serve as the unit of addition. In like manner, fractions may be added when they have a common unit; and since the fractional unit is the reciprocal of the denominator, fractions to be added must have a common denominator.

1 What is the sum of $\frac{a}{b}$ and $\frac{c}{b}$?

OPERATION.

$$\frac{a}{b}+\frac{c}{b}=\frac{a+c}{b}$$

ANALYSIS. The fractions have a common unit, $\frac{1}{b}$. In $\frac{a}{b}$ this unit is taken a times, and in $\frac{c}{b}$ it is taken c times; hence, in the sum of the fractions, it must be taken a plus c times, expressed $\frac{a+c}{b}$.

2. What is the sum of $\frac{a}{b}$, $\frac{c}{bn}$, and $\frac{d}{bm}$?

OPERATION.

$$\left.\begin{aligned}\frac{a}{b}&=\frac{amn}{bmn}\\ \frac{c}{bn}&=\frac{cm}{bmn}\\ \frac{d}{bm}&=\frac{dn}{bmn}\end{aligned}\right\}\text{Fractions reduced to a common denominator.}$$

ANALYSIS. We first reduce the given fractions to their least common denominator, and then add as in the first example.

$$\frac{amn}{bmn}+\frac{cm}{bmn}+\frac{dn}{bmn}=\frac{amn+cm+dn}{bmn}.$$

From these examples we derive the following

RULE. I. *Reduce the fractions to their least common denominator.*

II. *Add the numerators, and write the result over the common denominator.*

What is a fractional unit. Give analysis of Addition of Fractions. Rule.

NOTES. 1. When there are mixed quantities, the entire quantities and he fractions may be added separately; or the mixed quantities may be educed to fractions and added.

2. A fractional result should be reduced to its lowest terms.

EXAMPLES FOR PRACTICE.

3. What is the sum of $\frac{a}{3b}$ and $\frac{a+m}{2b}$? *Ans.* $\frac{5a+3m}{6b}$.

4. What is the sum of $\frac{x}{y}$, $\frac{z}{xy}$, and $\frac{y}{x}$? *Ans.* $\frac{x^2+z+y^2}{xy}$.

5. What is the sum of $\frac{a}{b}$ and $\frac{a+b}{c}$? *Ans.* $\frac{ac+ab+b^2}{bc}$.

6. What is the sum of $\frac{x}{2}$, $\frac{x}{3}$, and $\frac{x}{4}$? *Ans.* $x+\frac{x}{12}$.

7. What is the sum of $\frac{x-2}{3}$ and $\frac{4x}{7}$? *Ans.* $\frac{19x-14}{21}$.

8. What is the sum of $\frac{1}{a+b}$ and $\frac{1}{a-b}$? *Ans.* $\frac{2a}{a^2-b^2}$.

9. Add $\frac{x}{x+y}$ to $\frac{y}{x-y}$. *Ans.* $\frac{x^2+y^2}{x^2-y^2}$.

10. Add $\frac{12b-a}{35c}$ to $\frac{3a-b}{7c}$ *Ans.* $\frac{2a+b}{5c}$.

11. Add $\frac{1}{1+a}$, $\frac{a}{1-a}$, and $\frac{a}{1+a}$. *Ans.* $\frac{1}{1-a}$.

12. Add $\frac{a}{b}$, $\frac{a}{3b}$, and $\frac{5b}{4a}$. *Ans.* $\frac{16a^2+15b^2}{12ab}$.

13. Add $\frac{6ab-3b^2-12ac+16bc}{12bc}$ and $\frac{3a-4b}{3b}$.

Ans. $\frac{2a-b}{4c}$.

14. Add $2x$, $3x + \frac{3a}{5}$, and $x + \frac{2a}{9}$. *Ans.* $6x + \frac{37a}{45}$.

15. Add $5x + \frac{x-2}{3}$ and $4x + \frac{2x-3}{5x}$.

Ans. $9x + \frac{5x^2-4x-9}{15x}$.

16. Add $\frac{2b}{(a-b)(a+b)}$ and $\frac{1}{a+b}$. *Ans.* $\frac{1}{a-b}$.

17 Add $\frac{a-b}{ab}$, $\frac{b-c}{bc}$, and $\frac{c-a}{ac}$. *Ans.* 0.

18. Add $\frac{a^2-x^2}{ax}$ and $\frac{x-a}{x}$. *Ans.* $\frac{a-x}{a}$.

19. Add $\frac{5+x}{y}$, $\frac{3-ax}{ay}$, and $\frac{b}{3a}$. *Ans.* $\frac{15a+by+9}{3ay}$.

20. Add $\frac{a+b}{a-b}$ to $\frac{a-b}{a+b}$. *Ans.* $\frac{2(a^2+b^2)}{a^2-b^2}$.

21. Add $\frac{a}{b}$, $\frac{a-3b}{cd}$, and $\frac{a^2-b^2-ab}{bcd}$.

Ans. $\frac{acd-4b^2+a^2}{bcd}$.

22. Find the sum of $\frac{a}{a+b}$ and $\frac{b}{a-b}$. *Ans.* $\frac{a^2+b^2}{a^2-b^2}$.

23. Find the sum of $\frac{1}{x+y}$ and $\frac{y}{x^2-y^2}$. *Ans.* $\frac{x}{x^2-y^2}$.

24 Find the sum of $\frac{4a^2}{1-a^4}$ and $\frac{1-a^2}{1+a^2}$. *Ans.* $\frac{1+a^2}{1-a^2}$.

25. Find the sum of $\frac{1}{a} + \frac{1}{b}$ and $1 - \left(\frac{a+b}{ab}\right)$. *Ans.* 1.

SUBTRACTION.

117. We have seen (**54**), that one entire quantity may be subtracted from another, when they have a common factor to serve as the unit of subtraction. In like manner, one fraction may be subtracted from another when they have the same fractional unit, or a common denominator.

1. From $\frac{a}{b}$ subtract $\frac{c}{b}$.

OPERATION.

$$\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$$

ANALYSIS. The fractions have a common unit, $\frac{1}{b}$. In $\frac{a}{b}$, this unit is taken a times, and in $\frac{c}{b}$ it is taken c times; hence, in the difference of the fractions, it is taken a minus c times, expressed $\frac{a-c}{b}$.

From this example we derive the following

RULE. I. *Reduce the fractions to their least common denominator.*

II. *Subtract the numerator of the subtrahend from the numerator of the minuend, and write the result over the common denominator.*

NOTE.—Mixed quantities must be reduced to fractions before subtracting; and fractional results should be reduced to their lowest terms.

EXAMPLES FOR PRACTICE.

2 From $\frac{7x}{2}$ take $\frac{2x-1}{3}$.

Ans. $\frac{21x-4x+2}{6}=\frac{17x+2}{6}$.

Under what circumstances can one fraction be subtracted from another? Give analysis. Rule.

3. From $\frac{1}{x-y}$ take $\frac{1}{x+y}$.

OPERATION.

$$\frac{1}{x-y}-\frac{1}{x+y}=\frac{x+y}{x^2-y^2}-\frac{x-y}{x^2-y^2}=\frac{2y}{x^2-y^2}.$$

4. From $\frac{x}{3}$ take $\frac{2x}{7}$. *Ans.* $\frac{x}{21}$.

5. From $\frac{2ax}{3}$ take $\frac{5ax}{2}$. *Ans.* $-\frac{11ax}{6}$.

6. From $\frac{1}{a+1}$ take $\frac{a-2}{a^2-a+1}$. *Ans.* $\frac{3}{1+a^3}$.

7. From $\frac{3}{4a}$ take $\frac{5}{2x}$. *Ans.* $\frac{3x-10a}{4ax}$.

8. From $\frac{3a}{4x}$ take $\frac{4x}{3a}$. *Ans.* $\frac{9a^2-16x^2}{12ax}$.

9. From $\frac{1}{x-1}$ take $\frac{2}{x+1}$. *Ans.* $\frac{3-x}{x^2-1}$.

10. From $2a-2x+\frac{a-x}{a}$ take $2a-4x+\frac{x-a}{x}$.

Ans. $2x+\frac{a^2-x^2}{ax}$.

11. From $\frac{2a+b}{5c}$ take $\frac{3a-b}{7c}$. *Ans.* $\frac{12b-a}{35c}$.

12. From $\frac{5x+1}{7}$ take $\frac{21x+3}{4}$. *Ans.* $-\frac{127x+17}{28}$.

13. From $\frac{x-y}{2a}$ take $\frac{x+y}{3a}$. *Ans.* $\frac{ax-5ay}{6a^2}$.

14. From $\frac{1+a^2}{1-a^2}$ take $\frac{1-a^2}{1+a^2}$. *Ans.* $\frac{4a^2}{1-a^4}$.

15. From $x+\frac{x-y}{x^2+xy}$ take $\frac{x+y}{x^2-xy}$.

Ans. $x-\frac{4y}{x^2-y^2}$.

16. From $\frac{a-b}{2c}$ take $\frac{2b-4a}{5d}$.

Ans. $\frac{5ad-5bd-4bc+8ac}{10cd}$.

17. From $\frac{2(a^2+b^2)}{a^2-b^2}$ take $\frac{a-b}{a+b}$. *Ans.* $\frac{a+b}{a-b}$.

18. From $\frac{x}{x-3}$ take $\frac{x+3}{x}$. *Ans.* $\frac{9}{x^2-3x}$.

19. From $6a+\frac{14a-13}{20}$ take $4a+\frac{2a-5}{4}$.

Ans. $2a+\frac{a+3}{5}$.

20. What is the value of $\frac{a^2+b^2}{a^2-b^2}-\frac{b}{a-b}$?

Ans. $\frac{a}{a+b}$.

21. What is the value of $\frac{1+x^2}{1-x^2}-\frac{1-x^2}{1+x^2}$?

Ans. $\frac{4x^2}{1-x^4}$.

22. What is the value of $\frac{a-x}{a}-\frac{a^2-x^2}{ax}$?

Ans. $\frac{x-a}{x}$.

23. What is the value of $\frac{b+c}{bc}+\frac{c-a}{ac}-\frac{a-b}{ab}$?

Ans. $\frac{2}{a}$.

24. What is the value of $\frac{3x}{4}+\frac{2x}{5}-\frac{5x}{8}$? *Ans.* $\frac{21x}{40}$.

25. From $\frac{n-1}{n}$ take $\frac{n}{n-1}$. *Ans.* $\frac{1-2n}{n^2-n}$.

26. From $\frac{2}{1-x^2}$ take $\frac{1}{1-x}$. *Ans.* $\frac{1}{1+x}$.

27. From $\frac{a+c}{(a-b)(x-a)}$ take $\frac{b+c}{(a-b)(x-b)}$.

Ans. $\frac{x+c}{(x-a)(x-b)}$.

MULTIPLICATION.

CASE I.

118. To multiply a fraction by an entire quantity.

1. Multiply $\frac{a}{b}$ by c.

OPERATION.

$$\frac{a}{b} \times c = \frac{ac}{b}$$

ANALYSIS. A fraction may be multiplied by multiplying its numerator (**105**, I); we therefore multiply the numerator, a, by c, and obtain for the required product, $\frac{ac}{b}$.

2. Multiply $\frac{m}{xy}$ by x.

OPERATION.

$$\frac{m}{xy} \times x = \frac{m}{y}$$

ANALYSIS. A fraction may be multiplied by dividing its denominator (**105**, II); we therefore divide the denominator, xy, by x, and obtain for the required product, $\frac{m}{y}$. Hence,

Multiplying a fraction consists in multiplying its numerator, or dividing its denominator.

EXAMPLES FOR PRACTICE.

3. Multiply $\frac{c}{d}$ by m. *Ans.* $\frac{cm}{d}$.

4. Multiply $\frac{a^2x}{c-d}$ by ax. *Ans.* $\frac{a^3x^2}{c-d}$.

5. Multiply $\frac{mz}{c^2d}$ by cd. *Ans.* $\frac{mz}{c}$.

6. Multiply $\frac{4x}{21}$ by 7. *Ans.* $\frac{4x}{3}$.

7. Multiply $\frac{a-x}{x-1}$ by $a+x$. *Ans.* $\frac{a^2-x^2}{x-1}$.

8. Multiply $\frac{ac}{b(x+y)}$ by $x+y$. *Ans.* $\frac{ac}{b}$.

What is Case I? Give analysis. Deduction.

9. Multiply $\frac{cd}{m^2 - y^2}$ by $m - y$. *Ans.* $\frac{cd}{m + y}$.

10. Multiply $\frac{m}{x^5 - x}$ by $x^2 + 1$. *Ans.* $\frac{m}{x^3 - x}$.

119. It is usually advantageous to indicate the multiplication, and apply cancellation before obtaining the actual product.

1. Multiply $\frac{a}{x}$ by x.

OPERATION.

$$\frac{a \times x}{x} = a$$

ANALYSIS. Having indicated the multiplication, we cancel the common factor, x, from both numerator and denominator, and we have a for the product.

2. Multiply $\frac{c}{3m}$ by $6m$.

OPERATION.

$$\frac{c \times 6m}{3m} = 2c$$

ANALYSIS. Having indicated the multiplication, we cancel $3m$, and obtain $2c$ for the product. Hence,

I. *A fraction is multiplied by its own denominator by simply suppressing the denominator.*

II. *If a fraction be multiplied by its own denominator, or by any multiple of that denominator, the product will be an entire quantity.*

EXAMPLES FOR PRACTICE.

3. Multiply $\frac{x}{y}$ by y. *Ans.* x.

4. Multiply $\frac{3ax}{5b}$ by $5b$. *Ans.* $3ax$.

5. Multiply $\frac{cd^2}{a - x}$ by $a - x$. *Ans.* cd^2.

When may cancellation be applied in multiplication? Give first deduction. Second.

6. Multiply $\frac{3a^2}{x}$ by $2x^3$. *Ans.* $6a^2x^2$.

7. Multiply $\frac{3a-x}{10}$ by 20. *Ans.* $6a-2x$.

8. Multiply $\frac{a-x}{896}$ by 896. *Ans.* $a-x$.

9. Multiply $\frac{m^2}{a^2-x^2}$ by $a+x$. *Ans.* $\frac{m^2}{a-x}$.

10. Multiply $\frac{3c}{x-1}$ by x^2-1. *Ans.* $3c(x+1)$.

11. Multiply $\frac{a+b}{a-b}$ by $a^2-2ab+b^2$. *Ans.* a^2-b^2.

CASE II.

120. To multiply an entire or a fractional quantity by a fraction.

1. Multiply a by $\frac{b}{c}$.

FIRST OPERATION.

$$a \times \frac{b}{c} = \frac{b}{c} \times a = \frac{ab}{c}$$

ANALYSIS. It is evident that the product of two quantities is the same, whichever be taken as the multiplier; consequently a, multiplied by $\frac{b}{c}$, is equal to $\frac{b}{c}$ multiplied by a; and, according to Case I, $\frac{b}{c}$ multiplied by a, is $\frac{ab}{c}$.

SECOND OPERATION.

$$\frac{b}{c} = bc^{-1}$$

$$a \times \frac{b}{c} = abc^{-1} = \frac{ab}{c}$$

ANALYSIS. We first reduce the multiplier, $\frac{b}{c}$, to an entire form, bc^{-1} by (111). Then a, multiplied by bc^{-1} is abc^{-1}. which is equal to $\frac{ab}{c}$, (112), as before.

Give Case II. Analyses.

2. Multiply $\frac{a}{x}$ by $\frac{c}{m}$.

OPERATION.

$$\frac{a}{x} \times \frac{c}{m} = ax^{-1} \times cm^{-1} = acx^{-1}m^{-1} = \frac{ac}{xm}.$$

ANALYSIS. By (111), $\frac{a}{x}$ is equal to ax^{-1}, and $\frac{c}{m}$ is equal to cm^{-1}; and ax^{-1}, multiplied by cm^{-1}, is $acx^{-1}m^{-1}$, which is equal to $\frac{ac}{xm}$ (112). By inspecting this result we perceive that the numerator, ac, is the product of the given numerators, and the denominator, xm, is the product of the given denominators. Hence, the

RULE. I. *Reduce entire and mixed quantities to fractional forms.*

II. *Multiply the numerators together for a new numerator, and the denominators, for a new denominator, canceling all factors common to the numerator and denominator of the indicated product.*

EXAMPLES FOR PRACTICE.

3. Multiply $\frac{3a}{4x}$ by $\frac{5x}{8}$. *Ans.* $\frac{15a}{32}$.

4. Multiply $\frac{3a}{5y}$ by $\frac{3y}{9x}$. *Ans.* $\frac{a}{5x}$.

5. Multiply $\frac{3x^2}{10y}$ by $\frac{5y}{9x}$. *Ans.* $\frac{x}{6}$.

6. Multiply $\frac{a+m}{c^2}$ by $\frac{cx}{z}$. *Ans.* $\frac{x(a+m)}{cz}$.

7. Multiply $\frac{a-b}{5}$, $\frac{25x-25}{a^2-b^2}$, and $\frac{1}{x-1}$ together.

OPERATION.

$$\frac{a-b}{5} \times \frac{25(x-1)}{(a+b)(a-b)} \times \frac{1}{x-1} = \frac{5}{a+b}$$

Give analysis. Rule.

Analysis. We resolve the quantities into convenient factors, and *indicate* the multiplication. Since 5, $a - b$, and $x - 1$, are factors common to the numerator and denominator of the indicated product, we suppress or cancel them, and obtain for the result, $\frac{5}{a+b}$.

8. Multiply $\frac{x}{a+x}$, $\frac{a^2-x^2}{x^2}$, and $\frac{a}{a-x}$ together. *Ans.* $\frac{a}{x}$.

9. Multiply $\frac{3x}{2}$ by $\frac{3a}{b}$. *Ans.* $\frac{9ax}{2b}$.

10. Multiply $\frac{4a^2x}{3}$ by $\frac{3a}{4}$. *Ans.* a^3x.

11. Multiply $\frac{2x}{a}$ by $\frac{3ab}{c} \times \frac{3ac}{2b}$. *Ans.* $9ax$.

12. Multiply $-\frac{a}{x}$ by $-\frac{y}{z}$. *Ans.* $\frac{ay}{xz}$.

13. Multiply $\frac{a}{x}$, $\frac{3x}{y}$, $\frac{4y}{3z}$ together. *Ans.* $\frac{4a}{z}$.

14. Multiply $\frac{(a+x)}{30}$ by $\frac{5a}{3(a+x)}$. *Ans.* $\frac{a}{18}$.

15. Multiply $\frac{2x+3y}{2a}$ by $\frac{2a}{5x}$. *Ans.* $\frac{2x+3y}{5x}$.

16. What is the product of $\frac{4ax}{y}$, $\frac{3xy}{2a}$, and $\frac{2}{x}$? *Ans.* $12ax$.

17. What is the product of $\frac{2a}{3b+c}$ into $\frac{2ac-bc}{5ab}$?

Ans. $\frac{4ac-2bc}{15b^2+5bc}$.

18. Multiply $b + \frac{bx}{a}$ by $\frac{a}{x}$. *Ans.* $\frac{ab+bx}{x}$

19. Multiply $\frac{x^2-b^2}{bc}$ by $\frac{x^2+b^2}{b+c}$. *Ans.* $\frac{x^4-b^4}{b^2c+bc^2}$.

20. Multiply $\frac{a^2 - x^2}{2y}$ by $\frac{2a}{a + x}$. *Ans.* $\frac{(a - x)a}{y}$.

21. Multiply $\frac{x^2 - y^2}{x}$, $\frac{x}{x + y}$, and $\frac{a}{x - y}$. *Ans.* a.

22. Multiply $3a$, $\frac{x + 1}{2a}$, and $\frac{x - 1}{a + b}$ together.
Ans. $\frac{3(x^2 - 1)}{2(a + b)}$.

23. Multiply $\frac{3x^2 - 5x}{14}$ by $\frac{7a}{2x^3 - 3x}$. *Ans.* $\frac{3ax - 5a}{4x^2 - 6}$.

24. Multiply $\frac{3x^2}{5x - 10}$ by $\frac{15x - 30}{2x}$. *Ans.* $\frac{9x}{2}$.

25. Multiply $\frac{8ab}{3}$ by $\frac{3}{8ab}$. *Ans.* 1.

26. Multiply $\frac{x^2 - y^2}{ab}$ by $\frac{a^2}{x + y}$. *Ans.* $\frac{a(x - y)}{b}$.

27. Multiply $\frac{xyz}{x^4 + y^3}$ by $\frac{x^4 + y^3}{xyz}$. *Ans.* 1.

28. Multiply $\frac{a}{a - b}$ by $\frac{b}{a + b}$ *Ans.* $\frac{ab}{a^2 - b^2}$

29. What is the value of $\frac{(a - x)^3}{2a} \times \frac{3ab}{a - x} \times \frac{2c}{(a - x)^2}$?
Ans. $3bc$.

30. What is the value of $\frac{(x + y)^5}{(a + b)^4} \times \frac{(a + b)^5}{(x + y)^6}$?
Ans. $\frac{a + b}{x + y}$.

DIVISION.

CASE I.

121. To divide a fraction by an entire quantity.

1. Divide $\frac{ax}{b}$ by x.

OPERATION.

$$\frac{ax}{b} \div x = \frac{a}{b}$$

ANALYSIS. A fraction may be divided by dividing its numerator (105, I); we therefore divide the given numerator, ax, by x, and obtain for the required quotient, $\frac{a}{b}$.

2. Divide $\frac{m}{c}$ by a.

OPERATION.

$$\frac{m}{c} \div a = \frac{m}{ac}$$

ANALYSIS. A fraction may be divided by multiplying its denominator (105, II); we therefore multiply the denominator, c, by a, and obtain for the required quotient $\frac{m}{ac}$. Hence,

Dividing a fraction consists in dividing its numerator or multiplying its denominator.

EXAMPLES FOR PRACTICE.

3. Divide $\frac{3a^2x}{cd}$ by $3ax$. *Ans.* $\frac{a}{cd}$.

4. Divide $\frac{15ab^2}{4m^2}$ by $3b^2$. *Ans.* $\frac{5a}{4m^2}$.

5. Divide $\frac{a-x}{mz}$ by $2m$. *Ans.* $\frac{a-x}{2m^2z}$.

6. Divide $\frac{a}{x-y}$ by y. *Ans.* $\frac{a}{xy-y^2}$.

7. Divide $\frac{a^2-1}{cd}$ by $a+1$. *Ans.* $\frac{a-1}{cd}$.

8. Divide $\frac{m}{a^2-4}$ by a^2+4. *Ans.* $\frac{m}{a^4-16}$.

What is Case I? Give analyses. Deduction.

CASE II.

122. To divide an entire or a fractional quantity by a fraction.

1. Divide m by $\frac{a}{x}$.

OPERATION.

$$\frac{a}{x} = ax^{-1}.$$

$$m \div \frac{a}{x} = m \div ax^{-1} = \frac{m}{ax^{-1}} = \frac{mx}{a}.$$

ANALYSIS. We first reduce the divisor, $\frac{a}{x}$, to an entire form, ax^{-1}. Then m divided by ax^{-1} is $\frac{m}{ax^{-1}}$, which is equal to $\frac{mx}{a}$, (112).

2. Divide $\frac{a}{b}$ by $\frac{c}{d}$.

OPERATION.

$$\frac{a}{b} \div \frac{c}{d} = ab^{-1} \div cd^{-1} = \frac{ab^{-1}}{cd^{-1}} = \frac{ad}{bc}.$$

ANALYSIS. $\frac{a}{b}$ is equal to ab^{-1}, and $\frac{c}{d}$ is equal to cd^{-1}; and ab^{-1} divided by cd^{-1} is $\frac{ab^{-1}}{cd^{-1}}$, which is equal to $\frac{ad}{bc}$, (112). By inspecting this result, we perceive that the numerator, ad, is the numerator of the dividend multiplied by the denominator of the divisor; and the denominator, bc, is the denominator of the dividend multiplied by the numerator of the divisor. The same result can be more readily obtained by inverting the divisor, as in the second operation, and then multiplying the upper and the lower terms together. Hence, the following

SECOND OPERATION.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

RULE. I. *Reduce entire and mixed quantities to fractional forms.*

II. *Invert the terms of the divisor, and proceed as in multiplication.*

What is Case II.? Give analyses. Rule.

EXAMPLES FOR PRACTICE.

3 Divide $\frac{a}{1-a}$ by $\frac{a}{5}$. *Ans.* $\frac{5}{1-a}$.

4. Divide $\frac{2x}{ab}$ by $\frac{3xy}{ab}$. *Ans.* $\frac{2}{3y}$.

5. Divide $\frac{4a^2}{3mz}$ by $\frac{8ab}{m^2}$. *Ans.* $\frac{am}{6bz}$.

6 Divide $\frac{15ab}{a-x}$ by $\frac{10ac}{a^2-x^2}$. *Ans.* $\frac{3b(a+x)}{2c}$.

7. Divide $m+\frac{x}{y}$ by $\frac{c}{d}$. *Ans.* $\frac{d(my+x)}{cy}$.

8. Divide $a+\frac{ac}{x}$ by $\frac{ac}{x^2}$. *Ans.* $x+\frac{x^2}{c}$.

9. Divide $\frac{2ax+x^2}{a^3-x^3}$ by $\frac{x}{a-x}$. *Ans.* $\frac{2a+x}{a^2+ax+x^2}$.

10. Divide $\frac{14x-3}{5}$ by $\frac{10x-4}{25}$. *Ans.* $\frac{70x-15}{10x-4}$.

11. Divide $\frac{9x^2-3x}{5}$ by $\frac{x^2}{5}$. *Ans.* $\frac{9x-3}{x}$.

12. Divide $\frac{6x-7}{x-1}$ by $\frac{x+1}{3}$. *Ans.* $\frac{18x-21}{x^2-1}$.

13. Divide $\frac{16ax}{5}$ by $\frac{4x}{15}$. *Ans.* $12a$.

14. Divide $\frac{6z+4}{5}$ by $\frac{3z+2}{4y}$. *Ans.* $\frac{8y}{5}$.

15. Divide $\frac{7x}{3}$ by $\frac{4x^2}{6}$. *Ans.* $\frac{7}{2x}$.

16. Divide $\frac{a+1}{6}$ by $\frac{2a}{3}$. *Ans.* $\frac{a+1}{4a}$.

17. Divide $\frac{x}{x-1}$ by $\frac{x}{2}$. *Ans.* $\frac{2}{x-1}$.

18. Divide $\frac{x^2-2xy+y^2}{ab}$ by $\frac{x-y}{bc}$. *Ans.* $\frac{cx-cy}{a}$.

19. Divide $\frac{m^2 - n^2}{3}$ by $\frac{m+n}{6}$. *Ans.* $2m - 2n$.

20. Divide $\frac{5x}{3}$ by $\frac{2a}{3b}$. *Ans.* $\frac{5bx}{2a}$.

21. Divide $\frac{x-b}{8cd}$ by $\frac{3cx}{4d}$. *Ans.* $\frac{x-b}{6c^2x}$.

22. Divide $\frac{x^4 - b^4}{x^2 - 2bx + b^2}$ by $\frac{x^2 + bx}{x - b}$. *Ans.* $x + \frac{b^2}{x}$.

23. Divide $1 + \frac{1}{a}$ by $1 - \frac{1}{a^2}$. *Ans.* $\frac{a}{a-1}$.

24. Divide $\left(\frac{1}{1+x} + \frac{x}{1-x}\right)$ by $\frac{(1+x^2)^2}{(1-x^2)^2}$.

Ans. $\frac{1-x^2}{1+x^2}$.

25. Divide $\left(\frac{1}{a} + \frac{1}{ab^3}\right)$ by $\left(b + \frac{1}{b} - 1\right)$. *Ans.* $\frac{b+1}{ab^2}$.

123. The division of one fraction by another, or of one mixed quantity by another, may be *indicated* in the form of a *complex fraction*, and the result reduced to a *simple fraction.*

1. Divide $a + \frac{b}{c}$ *by* $x + \frac{y}{z}$.

OPERATION.

$$\frac{a + \frac{b}{c}}{x + \frac{y}{z}} = \frac{acz + bz}{cxz + cy}$$

ANALYSIS. We indicate the division by writing the dividend above a horizontal line, and the divisor below. Then, since the denominator of a fraction will disappear when the fraction is multiplied by any multiple of its denominator (119, II), we multiply both numerator and denominator of the complex fraction by cz, the least common multiple of the denominators of the fractional parts, and obtain the simple fraction, $\frac{acz + bz}{cxz + cy}$. Hence, to simplify a complex fraction,

Multiply both numerator and denominator by the least common multiple of the denominators of the fractional parts.

Explain the process of reducing a complex to a simple fraction. Give deduction.

EXAMPLES FOR PRACTICE.

2 Reduce $\dfrac{2\frac{3}{4}}{5\frac{1}{6}}$ to a simple fraction. *Ans.* $\frac{33}{62}$.

3. Reduce $\dfrac{a+\frac{m}{n}}{b-\frac{c}{d}}$ to a simple fraction. *Ans.* $\dfrac{adn+dm}{bdn-cn}$.

4. Reduce $\dfrac{a+\frac{a}{3}}{b}$ to a simple fraction. *Ans.* $\dfrac{4a}{3b}$.

5. Reduce $\dfrac{\frac{a}{4}+c}{x+\frac{z}{2}}$ to a simple fraction. *Ans.* $\dfrac{a+4c}{4x+2z}$.

6. Reduce $\dfrac{m}{a+\frac{1}{c}}$ to a simple fraction. *Ans.* $\dfrac{cm}{ac+1}$.

7. Simplify the fraction $\dfrac{a}{1+\frac{m}{n}}$. *Ans.* $\dfrac{na}{n+m}$.

8. Simplify the fraction $\dfrac{1-\frac{3}{a}}{1-\frac{1}{x}+\frac{1}{x^2}}$.

Ans. $\dfrac{(a-3)x^2}{a(x^2-x+1)}$.

9. Simplify the fraction $\dfrac{5c+\frac{a-b}{2x}}{5c-\frac{a-b}{2x}}$. *Ans.* $\dfrac{10cx+a-b}{10cx-a+b}$.

10. Simplify the fraction $\dfrac{\frac{m^2}{m^2-n^2}-1}{\frac{n^2}{m^2-n^2}+1}$. *Ans.* $\dfrac{n^2}{m^2}$.

11. Reduce $\dfrac{\frac{a^3-x}{2b^4}}{\frac{c+1}{2bx^2y^3}}$ to a simple fraction. *Ans.* $\dfrac{x^2y^3(a^3-x)}{b^3(c+1)}$.

SECTION II.

EQUATIONS.

124. An **Equation** is an expression of equality between two quantities; thus, $x = 4$, $5x = 60$, $3x = a + b$, are equations.

The **First Member** of an equation is the quantity on the left of the sign; and

The **Second Member** is the quantity on the right of the sign; thus, in the equation, $a + b = 7x - y$, the quantity, $a + b$, is the first member of the equation, and the quantity, $7x - y$, is the second member.

125. An **Arithmetical Equation** is one which expresses the equality of numbers or sets of numbers; as $10 = 10$; $4 + 3 = 6 + 1$.

126. An **Algebraic Equation** is one which contains one or more literal quantities; as, $3x = 12$; $c(a + b) = d$. Algebraic equations serve to express the relations between known and unknown quantities, and to determine the values of the unknown quantities by comparing them with some that are known.

127. A **Numeral Equation** is one in which all the known quantities are expressed by numbers; as, $3x + 2x = 25$.

128. A **Literal Equation** is one in which some or all the known quantities are expressed by letters; as, $x + 3cx = m$; $ay + x^2 = 91$.

129. An **Identical Equation** is one in which the two mem-

Define an equation. The members. An arithmetical equation. An algebraic equation. A numeral equation. A literal equation. An identical equation.

bers are the same, or capable of being reduced to the same expression by performing the operations indicated. Thus,

$$\left.\begin{array}{l} 2x - 1 = 2x - 1 \\ 5x + 3x = 8x \end{array}\right\} \text{ are identical equations.}$$

130. The **Degree** of an equation is denoted by the highest exponent of the unknown quantity in the equation. Thus,

$$\left.\begin{array}{r} x = a \\ x + bx = c \end{array}\right\} \text{ are equations of the first degree.}$$

$$\left.\begin{array}{r} x^2 + ax = c \\ ax^2 + bx = h \end{array}\right\} \text{ are equations of the second degree.}$$

$$\left.\begin{array}{r} x^3 = a \\ x^3 + bx = c \end{array}\right\} \text{ are equations of the third degree, \&c.}$$

131. A **Simple** equation is an equation of the first degree.

132. A **Quadratic** equation is an equation of the second degree.

133. A **Cubic** equation is an equation of the third degree.

TRANSFORMATION OF EQUATIONS.

134. The **Transformation** of an equation is the process of changing its form without destroying the equality of its members.

Since an equation is only an expression of equality between two quantities, all the changes that can be made in the members of an equation, by which their values are altered without destroying their equality, are embraced in the axioms (**46**), and may be stated as follows: —

I. *The same or equal quantities may be added to both members.* (Ax. 1.)

II. *The same or equal quantities may be subtracted from both members.* (Ax. 2.)

III. *Both members may be multiplied by the same or equal quantities.* (Ax. 3.)

Define the degree of an equation. A simple equation. A quadratic equation. A cubic equation. What is the transformation of an equation? State the principles upon which all transformations are based.

IV. *Both members may be divided by the same or equal quantities.* (Ax. 4.)

V. *Both members may be raised, by involution, to the same power.* (Ax. 8.)

VI. *Both members may be reduced, by evolution, to the same root.* (Ax. 9.)

NOTE.—As the principal object in transforming an equation is to find the value of the unknown quantity, we present here only those cases necessary to the solution of Simple Equations.

CASE I.

135. To transpose any term of an equation.

Transposition is the process of changing a term from one member of an equation to the other, without destroying the equality.

1. In $x + a = b$, transpose a to the second member.

OPERATION.

$$\begin{array}{lr} & x + a = b \\ \text{Subtract,} & a = a \\ \hline & x = b - a \end{array}$$

$$\begin{array}{lr} \text{Or,} & x + a = b \\ & x = b - a \end{array}$$

ANALYSIS. Since the equality of the members is not destroyed by taking the same quantity from both (**134**, II), we subtract a from each member, and obtain for a result, $x = b - a$.

The same result may be obtained by dropping $+ a$ from the first member, and writing $- a$ in the second member, as in the second operation.

2. In $b = x - c$, transpose c to the first member.

OPERATION.

$$\begin{array}{lr} & b = x - c \\ \text{Add,} & c = c \\ \hline & b + c = x \end{array}$$

$$\begin{array}{lr} \text{Or,} & b = x - c \\ & b + c = x \end{array}$$

ANALYSIS. We add c to both members of the equation (**134**, I), and obtain $b + c = x$. The same result may be obtained by dropping $- c$ from the second member and writing $+ c$ in the first member.

What is the object of transformation? What is Case I? Transposition? Give analyses.

It will be seen that in each of these examples, the term transposed disappears from the member in which it is given, and reappears in the other member with the opposite sign. Hence the following

RULE. *Drop the term to be transposed from the member in which it stands, and write it in the other member with its sign changed.*

EXAMPLES FOR PRACTICE.

In the following equations transpose the unknown terms to the first member, and the known terms to the second.

3. $3x + m = b$. *Ans.* $3x = b - m$.
4. $4x = 2x + cd$. *Ans.* $4x - 2x = cd$.
5. $3m + 12x - c = x + d$. *Ans.* $12x - x = d - 3m + c$.
6. $-5c^2d + ax = -bx - m$. *Ans.* $ax + bx = 5c^2d - m$.
7. $4acx - 3d^2 = a^2d - d^2x$. *Ans.* $4acx + d^2x = a^2d + 3d^2$.
8. $a + b - x - c^2 = 2c - bx$. *Ans.* $bx - x = 2c - a - b + c^2$.
9. $a^3x - cd = b + a^2x - ax$. *Ans.* $a^3x - a^2x + ax = b + cd$.
10. $ab - xc = bcd - g$. *Ans.* $-xc = bcd - g - ab$.
11. $m = ax - 3cx + m^2$. *Ans.* $3cx - ax = m^2 - m$.
12. $0 = ab - 3cx - 2ax - c$. *Ans.* $3cx + 2ax = ab - c$.

CASE II.

136. To clear an equation of fractions.

We have seen (**119**, II), that if a fraction be multiplied by any multiple of its denominator, the product will be an entir

Give Rule. Case II.

quantity. Hence it follows, that if several fractions be multiplied by a common multiple of their denominators, all the products will be entire quantities.

1. Transform $\frac{3x}{4} + \frac{x}{6} = 3$ into an equation having no fractional terms.

OPERATION.

$$\frac{3x}{4} + \frac{x}{6} = 3$$
$$9x + 2x = 36$$

ANALYSIS. We multiply every term of the equation by 12, the least common multiple of the denominators, canceling each denominator as we proceed; thus, 4 is contained in 12, 3 times, and 3 times $3x$ is $9x$; 6 is contained in 12 twice, and 2 times x is $2x$; and passing to the second member, 12 times 3 is 36. Hence, the

RULE. *Multiply all the terms of the equation by the least common multiple of the denominators, reducing each fractional term to an entire quantity by cancellation.*

NOTES. 1. If a fraction have the minus sign before it, change all the signs of the numerator, when the denominator disappears.

2. The pupil will readily see that an equation may also be cleared of fractions by multiplying each term by all the denominators.

EXAMPLES FOR PRACTICE.

Clear the following equations of their fractions: —

2. $\frac{x}{4} + \frac{2x}{3} = 11.$ *Ans.* $3x + 8x = 132.$

3. $\frac{5x}{6} - \frac{x}{4} = a.$ *Ans.* $10x - 3x = 12a.$

4. $\frac{x}{2} + \frac{x}{5} - \frac{2x}{10} = 3a - m.$
Ans. $5x + 2x - 2x = 30a - 10m.$

5. $\frac{x}{4} - \frac{x-3}{2} = \frac{a}{6}.$ *Ans.* $3x - 6x + 18 = 2a.$

6. $\frac{x}{4} + \frac{x}{8} + \frac{x}{6} = \frac{5}{12}.$ *Ans.* $6x + 3x + 4x = 10.$

Give analysis. Rule.

7. $\frac{x}{5} + a - 14 = \frac{x + a}{2}$.

Ans. $2x + 10a - 140 = 5x + 5a$.

8. $4 - \frac{x - a - d}{4} = \frac{1}{2}$. *Ans.* $16 - x + a + d = 2$.

9. $\frac{x - 4}{3} + \frac{2x}{7} = \frac{5}{14} - \frac{x - 3}{2}$.

Ans. $14x - 56 + 12x = 15 - 21x + 63$.

10. $\frac{x}{a} + \frac{x - 7}{2} = b - c$.

Ans. $2x + ax - 7a = 2ab - 2ac$.

11. $\frac{x}{a + b} + c = m$. *Ans.* $x + ac + bc = am + bm$.

12. $\frac{x}{a + b} - \frac{x}{a - b} = 4$.

Ans. $ax - bx - ax - bx = 4(a^2 - b^2)$.

13. $\frac{1}{bx} + \frac{1}{dx} + \frac{1}{fx} = 1$. *Ans.* $df + bf + bd = bdfx$.

14. $\frac{x}{a + b} + \frac{x}{a - b} = \frac{1}{a^2 - b^2}$.

Ans. $ax - bx + ax + bx = 1$.

15. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} + \frac{x}{5} = 77$.

Ans. $30x + 20x + 15x + 12x = 4620$.

16. $\frac{3x}{14} - \frac{x - 15}{21} = 20$. *Ans.* $9x - 2x + 30 = 840$.

17. $\frac{3x}{2} - \frac{5x}{3} + \frac{8x + 12}{7} = \frac{1}{14}$.

Ans. $63x - 70x + 48x + 72 = 3$.

18. $\frac{2x}{3a} - \frac{x - 1}{2ab} + \frac{3}{a^2} = 1 - x$.

Ans. $4abx - 3ax + 3a + 18b = 6a^2b(1 - x)$

19. $\frac{1 - \frac{x}{4} - m}{\frac{x}{6}} = 2$. *Ans.* $12 - 3x - 12m = 4x$.

20. $\frac{3x}{17} - 2\frac{1}{51} = 1\frac{1}{17}$. *Ans.* $9x - 103 = 54$.

REDUCTION OF SIMPLE EQUATIONS.

137. The **Reduction of an Equation** is the process of finding the value of the unknown quantity.

To reduce an equation, we must so transform it that the unknown quantity shall stand alone, and constitute one member; the other member will then be the value of the unknown quantity.

138. The value of the unknown quantity is said to be *verified*, when, being substituted for the unknown quantity, the two members of the equation prove to be equal. When this occurs, the *equation* also is said to be *satisfied*.

139. The unknown quantity of an equation may be united to known quantities in *four* different ways; by addition, by subtraction, by multiplication, and by division; and further by various combinations of *these four* ways, as shown by the following equations, both numeral and literal: —

	NUMERAL.	LITERAL.
1st. By addition, . .	$x + 6 = 10$	$x + a = b.$
2d. By subtraction, .	$x - 8 = 12$	$x - c = d.$
3d. By multiplication,	$20x = 80$	$ax = e.$
4th. By division, . .	$\frac{x}{4} = 16$	$\frac{x}{d} = g + a.$

5th. $x + 6 - 8 + 4 = 10 + 2 - 3$, $x + a - b + c = d + c$, &c., are equations in which the *unknown* is connected with known quantities, both by addition and subtraction.

6th. $2x + \frac{x}{3} = 21$, $ax + \frac{x}{b} = c$, are equations in which the *unknown* is connected with known quantities, by both multiplication and division.

What is the reduction of an equation? In what does it consist? The term, *verified*, is used to denote what? The term, *satisfied*, what? When and how is an equation reduced by addition? By subtraction? Multiplication? Division?

Equations often occur, in solving problems, in which all of these operations are combined.

140. When the unknown quantity is united to known quantities by addition or subtraction, it may be disunited, as we have seen, by transposition; which, in case of *plus* or *positive* terms, is equivalent to a process of *subtraction*, and in case of *minus* or *negative* terms, to a process of *addition*.

When the unknown quantity is united to others by *division*, it may be disunited or cleared of fractions, by *multiplication*; and, when united by *multiplication*, it may be disunited by *division*.

1. In the equation, $4x = 20$, what is the value of x?

OPERATION.

$$4x = 20$$
$$x = 5$$

VERIFICATION.

$$4 \times 5 = 20$$
$$20 = 20$$

ANALYSIS. Dividing both members of the equation by 4 (**134**, IV), we obtain $x = 5$. To *verify* this value of x, we multiply it by 4, the coefficient of x in the given equation, and obtain $20 = 20$, in which the members are equal; and the value is therefore verified, and the equation satisfied.

2. Given $x + \frac{x}{5} - \frac{x}{2} = 7$, to find the value of x.

OPERATION.

$$x + \frac{x}{5} - \frac{x}{2} = 7 \quad (1)$$
$$10x + 2x - 5x = 70 \quad (2)$$
$$7x = 70 \quad (3)$$
$$x = 10 \quad (4)$$

VERIFICATION.

$$10 + \frac{10}{5} - \frac{10}{2} = 7 \quad (1)$$
$$10 + 2 - 5 = 7 \quad (2)$$
$$7 = 7 \quad (3)$$

ANALYSIS. We clear the given equation of fractions, and obtain equation (2); uniting similar terms we have (3); and dividing both members by 7 we obtain $x = 10$.

To *verify* this value of x, we write the value, 10, instead of x, in the given equation, and obtain (1) for a result; performing the operations indicated, we have (2); and collecting the terms in the first member, we have $7 = 7$; and the value of x is verified, and the equation is satisfied.

8. Reduce the equation, $\frac{ax}{c} = b - x$.

OPERATION.

$$\frac{ax}{c} = b - x. \qquad (1)$$

$$ax = bc - cx. \qquad (2)$$

$$ax + cx = bc. \qquad (3)$$

$$(a + c)x = bc. \qquad (4)$$

$$x = \frac{bc}{a + c}. \qquad (5)$$

ANALYSIS. We first clear the equation of fractions, and obtain (2); transposing $-cx$, we have (3); factoring with reference to x, we have (4); and lastly, dividing both members of (4) by $a + c$, the coefficient of x, we have for the value of x, $\frac{bc}{a+c}$.

VERIFICATION.

$$\frac{a}{c} \times \frac{bc}{a + c} = b - \frac{bc}{a + c}. \qquad (1)$$

$$\frac{ab}{a + c} = \frac{ab}{a + c}. \qquad (2)$$

To verify this value, we substitute it for x in the given equation, and obtain (1); performing the operations indicated, we have (2), in which the two members are identical, and the value obtained for x is verified, and the equation is satisfied.

From these examples we deduce the following

RULE. To reduce an equation:—I. *Clear the equation of fractions, and perform all the operations indicated.*

II. *Transpose the unknown terms to the first member of the equation, and the known terms to the second member, and reduce each member to its simplest expression, factoring, when necessary, with reference to the unknown quantity.*

III. *Divide both members by the coefficient of the unknown quantity, and the second member will be the value required.*

To verify the result:—*Substitute the value found for the unknown quantity in the given equation, and perform the operations indicated. If the result is an identical equation, the value is verified.*

The three steps in the reduction of a simple equation, con-

Give the Rule for reducing an equation. For verifying the result. The steps in a reduction.

taining but one unknown quantity, may be briefly stated as follows: —

1st. Clear of fractions.
2d. Transpose and unite terms.
3d. Divide.

NOTES. 1. It is often advantageous to transpose and reduce, in part, before clearing of fractions.
2. In the last step, when the coefficient of the unknown quantity is negative, dividing will give a positive result.

EXAMPLES FOR PRACTICE.

4. Given $5x + 22 - 2x = 31$, to find x. *Ans.* $x = 3$.
5. Given $4x + 20 - 6 = 34$, to find x. *Ans.* $x = 5$.
6. Given $3x + 12 + 7x = 102$, to find x. *Ans.* $x = 9$.
7. Given $10x - 6x + 14 = 62$, to find x. *Ans.* $x = 12$.
8. Given $5x - 10 = 3x + 12$, to find x. *Ans.* $x = 11$.
9. Given $3x - 20 = -x - 4$, to find x. *Ans.* $x = 4$.
10. Given $4x + 45 = 7x - 30$, to find x. *Ans.* $x = 25$.
11. Given $x - 1 = 4x - 91$, to find x. *Ans.* $x = 30$.
12. Given $3(x + 1) + 4(x + 2) = 6(x + 3)$, to find x. *Ans.* $x = 7$.

NOTE. —First perform the multiplications indicated, and then reduce.

13. Given $5(x + 1) + 6(x + 2) = 6(x + 7)$, to find x. *Ans.* $x = 5$.
14. Give $7(x + 3) - 4(3x - 16) = 45$, to find x. *Ans.* $x = 8$.
15. Given $ax + bx = am + bm$, to find x.

OPERATION.

$$ax + bx = am + bm \quad (1)$$
$$(a + b)x = (a + b)m \quad (2)$$
$$x = m \quad (3)$$

ANALYSIS. We factor both members of the given equation and obtain equation (2); and dividing by $a + b$, the coefficient of x, we obtain $x = m$, the answer.

16. Given $cx - x = bc - b$, to find x. *Ans.* $x = b$.

17. Given $ax + dx = a - c$, to find x. *Ans.* $x = \frac{a-c}{a+d}$.

18. Given $ax + m = cx + n$, to find x. *Ans.* $x = \frac{n-m}{a-c}$.

19. Given $ax - bx = c + dx - m$, to find x.

Ans. $x = \frac{c-m}{a-b-d}$.

20. Given $\frac{3x}{4} + 16 = \frac{x}{2} + \frac{x}{8} + 17$, to find x.

OPERATION.

$$\frac{3x}{4} + 16 = \frac{x}{2} + \frac{x}{8} + 17 \quad (1)$$
$$\frac{3x}{4} = \frac{x}{2} + \frac{x}{8} + 1 \quad (2)$$
$$6x = 4x + x + 8 \quad (3)$$
$$x = 8 \quad (4)$$

ANALYSIS. We first drop 16 from both members of the equation, and obtain (2). Then, clearing of fractions, transposing and reducing, we have $x = 8$, the *Ans.*

21. Given $\frac{x}{2} - 3 + \frac{x}{3} = 5 - 3$, to find x. *Ans.* $x = 6$.

22. Given $\frac{x}{3} - \frac{x}{4} + 2 = 3$, to find x. *Ans.* $x = 12$.

23. Given $\frac{x}{4} + \frac{x}{8} - \frac{x}{6} = \frac{5}{12}$, to find x. *Ans.* $x = 2$.

24. Given $\frac{5x}{8} + \frac{1}{4} = \frac{11}{6} + \frac{7x}{12}$, to find x. *Ans.* $x = 38$.

25. Given $\frac{x}{a} + \frac{x-5}{2} + 2b = 3b$, to find x.

Ans. $\frac{2ab + 5a}{2 + a}$.

26. Given $\frac{3x}{5} + 2\frac{1}{2} + 11 = \frac{x}{4} + 17$, to find x.

Ans. $x = 10$.

27. Given $\frac{1}{2}x + \frac{1}{3}x + \frac{1}{4}x = 39$, to find the value of x.

OPERATION.

$$\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=39 \quad (1)$$
$$6x+4x+3x=39\times 12 \quad (2)$$
$$13x=39\times 12 \quad (3)$$
$$x=3\ \times 12 \quad (4)$$
$$\text{or,}\quad x=36 \quad (5)$$

ANALYSIS. We multiply both members by 12, the least common multiple of the denominators, *indicating* the operation in the second member, and obtain (2); reducing, we have (3); we next divide by 13, observing, in the second member, that a quantity may be divided by dividing any one of its factors, and obtain (4); whence $x=36$.

28. Given $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x+\frac{1}{5}x=77$, to find x.
Ans. $x=60$.

29. In $\frac{1}{2}x+\frac{1}{3}x+\frac{1}{4}x=130$, find x. *Ans.* $x=120$.

30. Given $\frac{1}{2}x+\frac{1}{6}x+\frac{1}{12}x=90$, to find x.
Ans. $x=120$.

31. Given $\frac{1}{2}y+\frac{1}{3}y+\frac{1}{7}y=82$, to find y.
Ans. $y=84$.

32. Given $5x+\frac{1}{6}x+\frac{1}{2}x=34$, to find x.
Ans. $x=6$.

33. Given $\frac{3x}{4}-\frac{x-1}{2}=6x-\frac{20x+13}{4}$, to find x.

FIRST OPERATION.

$$\frac{3x}{4}-\frac{x-1}{2}=6x-\frac{20x+13}{4} \quad (1)$$
$$3x-2x+2=24x-20x-13 \quad (2)$$
$$-3x=-15 \quad (3)$$
$$x=5 \quad (4)$$

ANALYSIS. We multiply by 4 to clear of fractions. The first term multiplied by 4 is $3x$; the product of the second term is $2x-2$; but since the fraction has the *minus sign* before it, this quantity must be *subtracted*, which is done by changing the signs of the terms, thus $-2x+2$. The product of the first term in the second member is $24x$; the product of the second term, or fraction, in the second member is $20x+13$, which being *subtracted*, as indicated by the *sign* before the fraction, becomes $-20x-13$. Reducing, we obtain $x=5$.

NOTE.—Young operators are liable to the mistake of omitting to change he signs of all the terms, when a fraction having the minus sign before t, and a polynomial numerator, is reduced to an entire quantity. This error may be avoided by the method which appears below.

SECOND OPERATION.

$$\frac{3x}{4} - \frac{x-1}{2} = 6x - \frac{20x+13}{4} \quad (1)$$

$$\frac{3x}{4} + \frac{20x+13}{4} = 6x + \frac{x-1}{2} \quad (2)$$

$$3x + 20x + 13 = 24x + 2x - 2 \quad (3)$$

$$-3x = -15 \quad (4)$$

$$x = 5 \quad (5)$$

ANALYSIS. We first *transpose* the fractions having the minus sign, and obtain (2) in which the fractions are positive; then clearing of fractions and reducing, we have $x = 5$ as before.

34. Given $x - \frac{x-3}{2} = \frac{9}{2} - \frac{x+4}{3}$, to find x.

Ans. $x = 2$.

35. Given $\frac{x+2}{3} - \frac{x-3}{4} + 2 = x - \frac{x-1}{2}$, to find x.

Ans. $x = 7$.

36. Given $\frac{4x-2}{11} - \frac{3x-5}{13} = 1$, to find x. *Ans.* $x = 6$.

37. Given $\frac{x}{5} - \frac{x-2}{3} = -\frac{x}{2} + \frac{13}{3}$, to find x.

Ans. $x = 10$.

Find the value of the unknown quantity in each of the following equations:

38. $\frac{x}{2} + \frac{x}{3} + \frac{x}{4} = a$. *Ans.* $x = \frac{12a}{13}$.

39. $\frac{x}{a} + \frac{x}{b} + \frac{x}{c} = 1$. *Ans.* $x = \frac{abc}{ac + bc + ab}$.

40. $\frac{1+x}{1-x} = \frac{1+a}{a}$. *Ans* $x = \frac{1}{2a+1}$.

41. $\frac{x+3}{4} - \frac{x-8}{5} - \frac{x-5}{2} = -1.$ *Ans.* $x = 13.$

42. $\frac{x}{8} - 5 + \frac{x}{4} + 8 + \frac{x}{5} - 10 = 100 - 13.$

Ans. $x = 120.$

43. $\frac{15}{x+3} - \frac{1}{3x+9} = 2\frac{14}{15}.$ *Ans.* $x = 2.$

44. $\frac{a^2}{x} + \frac{b^2}{x} + \frac{c^2}{x} = 2.$ *Ans.* $x = \frac{1}{2}(a^2 + b^2 + c^2).$

45. $(a+b)(a-b)(x-m) = b^2m.$ *Ans.* $x = \frac{a^2m}{a^2 - b^2}.$

46. $\frac{11x - 80}{6} - \frac{8x - 5}{15} = 0.$ *Ans.* $x = 10.$

47. $\frac{1 + \frac{x}{4} + m}{\frac{x}{8}} = \frac{12(m+6)}{2x}.$ *Ans.* $x = 20.$

48. $x - 3 + 2(x - 3) = \frac{x-3}{3} - \frac{x-3}{4} + a.$

OPERATION.

Put $x - 3 = y;$ (1)

Then $y + 2y = \frac{y}{3} - \frac{y}{4} + a$ (2)

$y = \frac{12a}{35}$ (3)

Or, $x - 3 = \frac{12a}{35}$ (4)

$x = 3 + \frac{12a}{35}$ (5)

ANALYSIS. To simplify the equation, we represent $x - 3$ by y, and the equivalent equation is (2), reducing which, we obtain $y = \frac{12a}{35}$. But y represents $x - 3$; and restoring this value in (1), we obtain (4), whence, by transposing, $x = 3 + \frac{12a}{35}$.

49. $\frac{x+5}{2} + \frac{5(x+5)}{6} = 3(x+5) - 20.$ *Ans.* $x = 7.$

50. $x - a + \frac{x-a}{2} + \frac{3(x-a)}{4} = 2\frac{1}{4}.$ *Ans.* $x = a + 1.$

51. Given $\frac{9x + 20}{36} = \frac{4x - 12}{5x - 4} + \frac{x}{4}$, to find x.

OPERATION.

$$\frac{9x+20}{36}=\frac{4x-12}{5x-4}+\frac{x}{4} \quad (1)$$

$$20=\frac{36(4x-12)}{5x-4} \quad (2)$$

$$20(5x-4)=36(4x-12) \quad (3)$$

$$5(5x-4)=9(4x-12) \quad (4)$$

$$-11x=-88 \quad (5)$$

$$x=8 \quad (6)$$

ANALYSIS. We first multiply by 36, and dropping $9x$ from both members, we obtain (2). Next, clearing of fractions, we obtain (3); dividing by 4, we obtain (4); performing the multiplication indicated, and reducing, we obtain (5) and (6).

52. Given $\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3}$, to find x.

OPERATION.

$$\frac{6x+7}{9}+\frac{7x-13}{6x+3}=\frac{2x+4}{3} \quad (1)$$

$$\frac{6x+7}{3}+\frac{7x-13}{2x+1}=2x+4 \quad (2)$$

$$\frac{21x-39}{2x+1}=5 \quad (3)$$

$$21x-39=10x+5 \quad (4)$$

$$11x=44 \quad (5)$$

$$x=4 \quad (6)$$

ANALYSIS. We first multiply each term by 3, which is done by dividing each denominator by 3, and obtain (2); multiplying again by 3, and reducing, we obtain (3); clearing of fractions and reducing, we obtain, finally, (6).

NOTE.—By clearing equations of the simplest denominators first, as in the examples just given, we sometimes avoid not only laborious multiplications, but the involution of the unknown quantity to higher powers.

53. Given $\frac{7x+16}{21}=\frac{x+8}{21}+\frac{x}{3}$, to find x.

Ans. $x=8$.

54. Given $\frac{7x+2a}{21}=\frac{x+a}{4x-11}+\frac{x}{3}$, to find x.

Ans. $x=\frac{43a}{8a-21}$.

55. Given $\frac{2x+1}{29}-\frac{402-3x}{12}=9-\frac{471-6x}{2}$, to find x.

Ans. $x=72$.

56. Given $\frac{18x-19}{28}+\frac{11x+21}{6x+14}=\frac{9x+15}{14}$, to find x.

Ans. $x=7$

57. Given $\frac{1}{x-5}+5+x=\frac{3x+18}{3}$, to find x.

Ans. $x=6$.

58. Given $\frac{m(3a^2-2b^2)}{a+b}=\frac{2a^2-b^2}{ax+bx}+\frac{a-b}{x}$, to find x.

Ans. $x=\frac{1}{m}$.

PROPORTION.

141. It is often convenient to express the relations of algebraic quantities in the form of a proportion, and from the proportion derive an equation.

For this reason we present here so much of the theory of proportion as will enable the pupil to make use of this practical advantage, reserving the full discussion of the subject for a subsequent chapter.

142. Ratio is the quotient of one number divided by another. Thus, the ratio of A to B is $\frac{B}{A}$.

143. Proportion is the equality of ratios.

Thus, if $\frac{B}{A}=r$, or $B=rA$;

and $\frac{D}{C}=r$, or $D=rC$;

then the four quantities, A, B, C, and D, are proportional, and their proportionality is expressed thus:

$$A:B::C:D,$$

in which A and D are called the *extremes*, and B and C the *means*.

Define Ratio. Proportion. The extremes. The means.

If in the place of B and D we write their values, rA and rC, this proportion will be

$$A : rA :: C : rC; \text{ and we have}$$

$A \times rC = rAC$, the product of the extremes; and
$C \times rA = rAC$, the product of the means. Hence,

The product of the extremes is equal to the product of the means.

1 Given $2 : x :: 6 : 5x - 4$, to find x.

OPERATION.

$$2 : x :: 6 : 5x - 4$$
$$10x - 8 = 6x \quad (1)$$
$$4x = 8 \quad (2)$$
$$x = 2 \quad (3)$$

ANALYSIS. The extremes are 2 and $5x - 4$, and their product is $10x - 8$; the means are x and 6, and their product is $6x$; and since these products are equal, we have equation (1). Reducing, we have $x = 2$.

Hence, to convert a proportion into an equation, we have the following

RULE. *Place the product of the extremes equal to the product of the means.*

EXAMPLES FOR PRACTICE.

2. Given $x : 25 :: 60 : 3$, to find x. *Ans.* $x = 500$.
3. Given $x : x + 6 :: 2 : 6$, to find x. *Ans.* $x = 3$.
4. Given $x + 2 : a :: b : c$, to find x. *Ans.* $x = \frac{ab}{c} - 2$.
5. Given $x + 6 : 38 - x :: 9 : 2$, to find x. *Ans.* $x = 30$.
6. Given $x + 4 : x - 11 :: 100 : 40$, to find x. *Ans.* $x = 21$.
7. Given $x + a : x - a :: c : d$, to find x. *Ans.* $x = \frac{a(c + d)}{c - d}$.

Rule for changing a proportion to an equation.

8. Given $x : 2x - a :: a : b$, to find x. *Ans.* $x = \frac{a^2}{2a - b}$.

9. Given $a : b :: 2y : d$, to find y. *Ans.* $y = \frac{ad}{2b}$.

10. Given $a^2 - ac : ax :: 1 : (d - b)$, to find x.
Ans. $x = (d - b)(a - c)$.

11. Given $x : 75 - x :: 3 : 2$, to find x. *Ans.* $x = 45$.

PROBLEMS.

144. A **Problem** is a question requiring the values of unknown quantities from given conditions.

145. The **Solution of a Problem** is the process of finding the values of the unknown quantities.

We have seen that equations serve to express the relations of algebraic quantities, and to determine the values of the unknown. In Algebra, the solution of problems is generally effected by means of equations.

1. If from five times a certain number, 24 be subtracted, the remainder will be equal to 16; required the number.

SOLUTION.

Let $x =$ the number.

$$5x - 24 = 16 \quad (1)$$
$$5x = 40 \quad (2)$$
$$x = 8 \quad (3)$$

ANALYSIS. We denote the number required by x. By the condition of the problem, 5 times x minus 24 is equal to 16, which, expressed algebraically, gives equation (1).

Reducing this equation, we have $x = 8$, the number required.

In this problem, the condition from which the equation is formed is clearly expressed, and furnishes the equation *directly*.

2. A merchant paid \$480 to two men, A and B, and he paid three times as much to B as to A. How many dollars did he pay to each?

Define a problem. The solution of a problem.

SOLUTION.

Let $x =$ the sum paid to A.
$3x =$ the sum paid to B.
$4x =$ the sum paid to both.

$$4x = 480. \quad (1)$$
$$x = 120, \text{ A's share.} \quad (2)$$
$$3x = 360, \text{ B's share.} \quad (3)$$

ANALYSIS. We let x represent the sum paid to A; and since he paid 3 times as much to B as to A, $3x$ will represent the sum paid to B; and by addition, $4x$ is the sum paid to both. But by one condition of the problem, 480 dollars is the sum paid to both. Therefore $4x$ is equal to 480, which expressed algebraically, is equation (1). Reducing, we find the value of x, or the sum paid to A, to be $120; and 3 times this sum, or 3 times the value of x, the sum paid to B, $360.

In this problem the equation is drawn from an *implied* condition. In the algebraic notation, $4x$ is the sum paid to both; and in the enunciation of the problem, 480 dollars is the sum paid to both; these two expressions of the same quantity are put equal to each other, to form the equation.

From the examples given, we derive the following

GENERAL RULE.

I. *Represent one of the quantities whose value is to be determined by some letter or symbol, and from the known relations find an algebraic expression for each of the other quantities, if any, involved in the question.*

II. *Form an equation from some condition expressed or implied, by indicating the operations necessary to verify the value of the unknown quantity.*

III. *Reduce the equation.*

The three parts of the rule, or the three steps in the solution of a problem, may be named as follows:

1st. The notation;

2d. The equation;

3d. The reduction of the equation.

NOTE.—By the first two steps, the problem is translated from *common* into *algebraic* language; and this is called the *statement* of the problem.

Give general Rule for solution of problems. Steps of the process.

PROBLEMS.

3. A father is 3 times as old as his son, and the difference of their ages is 24 years; what is the age of each?

Ans. Son's age, 12; father's age, 36.

4. A gentleman purchased a horse, a chaise and a harness, for $230. The chaise cost 3 times as much as the harness, and the horse $20 more than the chaise; what was the cost of each?

Ans. Harness, $30. Chaise, $90. Horse, $110.

5. Two men bought a carriage for 86 dollars; one paid 26 dollars more than five times as much as the other; what did each pay? *Ans.* One paid 10, the other 76 dollars.

6. A man had six sons, to whom he gave 120 dollars, giving to each one 4 dollars more than to his next younger brother; how many dollars did he give to the youngest?

Ans. $10.

7. Three men received 65 dollars, the second receiving 5 dollars more than the first, and the third 10 dollars more than the second; what sum did the first receive? *Ans.* $15.

8. A man paid a debt of 29 dollars, in three different payments; the second payment was 3 dollars more than at first, and the third payment was twice as much as the second; what was the amount of the first payment? *Ans.* $5.

9. The greater of two numbers exceeds the less by 14, and 3 times the greater is equal to 10 times the less; what are the numbers? *Ans.* 6 and 20.

10. Moses is 16 years younger than his brother Joseph, but 3 times the age of Joseph is equal to 5 times that of Moses; what are their ages? *Ans.* 24 and 40.

11. On a certain day, a merchant paid out $2500 to three men, A, B, and C; he paid to B $500 less than the sum paid to A, and to C $900 more than to A; required the sum paid to A. *Ans.* $700.

12. There are three numbers which together make 72; the second is twice as much as the first, and the third is as much as both the others; what are the numbers?

Ans. 1st is 12; 2d, 24; 3d, 36.

13. A man paid \$750 to two creditors, A and B, paying 4 times as many dollars to B as to A; how much did he pay to each? *Ans.* To A, \$150; to B, \$600.

The last problem would have been essentially the same in character, if any other sum had been paid out, and if the money had been distributed to the two men in any other ratio. We may therefore make this problem general by stating it as follows:

A merchant paid a dollars to two men, A and B, paying n times as many dollars to B as to A; how much did he pay to each?

SOLUTION.

Let $x =$ the sum paid to A;
$nx =$ the sum paid to B.

$$x + nx = a \qquad (1)$$

$$(1 + n)x = a \qquad (2)$$

$$x = \frac{a}{1 + n}, \text{ the sum to A}; \qquad (3)$$

$$nx = \frac{na}{1 + n}, \text{ the sum to B}. \qquad (4)$$

ANALYSIS. Let x denote the sum paid to A; then nx will denote the sum paid to B, and $x + nx$ the sum paid to both, which, by the condition of the problem, must be equal to a, as expressed in (1). Reducing, we obtain the value of x, or A's part, in (3); and multiplying by n we obtain the value of nx, or B's part, in (4).

VERIFICATION.

$$\frac{a}{1+n} + \frac{na}{1+n} = \frac{a + na}{1 + n} = \frac{(1 + n)a}{1 + n} = a$$

To verify these values, we add them, and find that their sum is a, the whole amount paid to both, as stated in the problem.

14. My horse and saddle are worth \$100, and my horse is worth 7 times as much as my saddle; what is the value of each? *Ans.* Saddle, \$12½; horse, \$87½.

15. My horse and saddle are worth a dollars, and my horse is worth n times as much as my saddle; what is the value of each? *Ans.* Saddle, $\frac{a}{1+n}$; horse, $\frac{na}{1+n}$.

16. A farmer had 4 times as many cows as horses, and 5 times as many sheep as cows, and the number of them all was 100; how many horses had he? *Ans.* 4.

17. A farmer had n times as many cows as horses, and m times as many sheep as cows, and the number of them all was a; how many horses had he? *Ans.* $\frac{a}{1+n+mn}$ horses.

18. A school-girl had 120 pins and needles, and she had seven times as many pins as needles; how many had she of each sort? *Ans.* 15 needles, and 105 pins.

19. A teacher said that her school consisted of 64 scholars, and that there were 3 times as many in Arithmetic as in Algebra, and 4 times as many in Grammar as in Arithmetic; how many were there in each study?

Ans. 4 in Algebra; 12 in Arithmetic; and 48 in Grammar.

20. There was a school consisting of a scholars; a certain portion of them studied Algebra, n times as many studied Arithmetic, and there were m times as many in Grammar as in Arithmetic; how many were in Algebra?

Ans. $\frac{a}{1+n+mn}$.

21. A person was \$450 in debt. He owed to A a certain sum, to B twice as much as to A, and to C twice as much as to A and B; how much did he owe each?

Ans. To A, \$50; to B, \$100; to C, \$300.

22. A person said that he was owing to A a certain sum, to B four times as much, to C eight times as much, and to D six times as much; and that \$570 would make him even with the world; what was his debt to A? *Ans.* \$30.

23. A person said that he was in debt to four individuals, A, B, C, and D, to the amount of a dollars; and that he was indebted to B, n times as many dollars as to A; to C, m times as many dollars as to A; and to D, p times as many dollars as to A; what was his debt to A?

Ans. $\dfrac{a}{1+n+m+p}$ dollars.

24. If \$75 be divided between two men in the proportion of 3 to 2, what will be the respective shares?

SOLUTION.

Let $x =$ the greater share;
$75 - x =$ the smaller share;

$$x : 75 - x :: 3 : 2$$
$$2x = 225 - 3x \quad (1)$$
$$5x = 225 \quad (2)$$
$$x = 45\text{, greater;} \quad (3)$$
$$75 - x = 30\text{, smaller.} \quad (4)$$

ANALYSIS. We express the greater share by x, and the smaller share by $75 - x$. By the conditions of the problem, these shares are in the proportion of 3 to 2, as expressed in the proportion. Converting this proportion into an equation and reducing, we find the respective shares to be \$45 and \$30.

SECOND SOLUTION.

Let $3x =$ the greater share;
$2x =$ the smaller share;

$$5x = 75 \quad (1)$$
$$x = 15 \quad (2)$$
$$3x = 45\text{, greater;} \quad (3)$$
$$2x = 30\text{, smaller.} \quad (4)$$

ANALYSIS. Since the shares are in the proportion of 3 to 2, they may be represented by $3x$ and $2x$. The sum of the shares, $5x$, must be equal to \$75, the amount divided; whence, by reduction, we find the shares to be \$45 and \$30. as before.

NOTE.—When proportional numbers are required, it is generally most convenient to represent them by only one letter, with coefficients of the given relation or proportion. Thus, numbers in proportion of 3 to 4, may be expressed by $3x$ and $4x$, and the proportion of a to b may be expressed by ax and bx. This avoids the use of a proportion in the solution of the problem.

25. Divide \$150 into two parts, so that the smaller may be to the greater as 7 to 8. *Ans.* \$70, and \$80.

26. Divide \$1235 between A and B, so that A's share may be to B's as 3 to 2. *Ans.* A's share, \$741; B's, \$494.

27. Divide d dollars between A and B, so that A's share may be to B's as m is to n.

Ans. A's share, $\frac{md}{m+n}$; B's, $\frac{nd}{m+n}$.

28. Two men commenced trade together; the first put in \$40 more than the second, and the stock of the first was to that of the second as 5 to 4; what was the stock of each? *Ans.* \$200, and \$160.

29. A man was hired for a year, for \$100 and a suit of clothes; but at the end of 8 months he left, and received his clothes and \$60 in money, as full compensation for the time he had worked; what was the value of the suit of clothes? *Ans.* \$20.

30. Three men trading in company gained \$780, which must be divided in proportion to their stock. A's stock was to B's as 2 to 3, and A's to C's as 2 to 5. What part of the gain must each receive?

Ans. A, \$156; B, \$234; C, \$390.

31. A field of 864 acres is to be divided among three farmers, A, B, and C, so that A's part shall be to B's as 5 to 11, and C may receive as much as A and B together; how much must each receive?

Ans. A, 135 acres; B, 297; and C, 432.

32. Three men trading in company, put in money in the following proportion; the first, 3 dollars as often as the second 7, and the third 5. They gain \$960. What is each man's share of the gain? *Ans.* \$192; \$448; \$320.

33. A man had two flocks of sheep, each containing the same number; from one he sold 80, and from the other 20; then the number remaining in the former was to the number in the latter as 2 to 3. How many sheep did each flock originally contain? *Ans.* 200.

34. A gentleman is now 25 years old, and his youngest brother is 15. How many years must elapse before their ages will be in the proportion of 5 to 4? *Ans.* 25 years.

35. There are two numbers in the proportion of 3 to 4; but if 24 be added to each of them, the two sums will be in the proportion of 4 to 5. What are the numbers?

Ans. 72 and 96.

36. A man's age when he was married was to that of his wife as 3 to 2; and when they had lived together 4 years, his age was to hers as 7 to 5. What were their ages when they were married? *Ans.* His age, 24 years; hers, 16.

37. The difference between two numbers is 12, and the greater is to the less as 11 to 7; what are the numbers?

Ans. 21 and 33.

38. The difference between two numbers is a, and the greater is to the less as m to n; what are the numbers?

Ans. $\dfrac{ma}{m-n}$ and $\dfrac{na}{m-n}$.

39. The sum of two numbers is 20, and their sum is to their difference as 10 to 1; what are the numbers?

Ans. 9 and 11.

40. The sum of two numbers is a, and their sum is to their difference as m to n; what are the numbers?

Ans. Greater, $\dfrac{(m+n)a}{2m}$; less, $\dfrac{(m-n)a}{2m}$

41. A certain sum of money was put at simple interest, and

in 8 months it amounted to \$120; had the interest continued 14 months, the amount would have been \$126. What was the principal?

SOLUTION.

Let $x =$ the principal.
$120 - x =$ int. for 8 mo.
$126 - x =$ int. for 14 mo.

$$120 - x : 126 - x :: 8 : 14 \quad \text{(A)}$$

Put $120 = a$; then

$$a - x : a + 6 - x :: 8 : 14 \quad \text{(B)}$$

$$8a + 48 - 8x = 14a - 14x \quad (1)$$

$$6x = 6a - 48 \quad (2)$$

$$x = a - 8 \quad (3)$$

$$\text{or } x = 112 \quad (4)$$

ANALYSIS. We let x represent the principal, and, subtracting it from each amount, we have $120 - x$, the interest for 8 months, and $126 - x$, the interest for 14 months. But since interest is proportional to the time, we have proportion (A). For brevity, we write a for 120, and obtain (B). Converting this proportion into an equation, and reducing, we have $x = a - 8$, or $120 - 8$, or 112.

NOTE.—The artifice, employed above, of representing a numeral by a letter, and restoring the value in the final result, is of much use, and gives true delicacy to algebraic operations. The pupil should be encouraged in its use.

42. A sum of money placed at simple interest, in 13 months amounted to \$113, and in 20 months, to \$120. Required, the sum at interest. *Ans.* \$100.

43. A certain number diminished by 45, is to the same number increased by 45, as 1 to 31. What is the number? *Ans.* 48.

44. The number 12 is $\frac{3}{4}$ of what number?

SOLUTION.

Let $x =$ the number.

$$\frac{3x}{4} = 12 \quad (1)$$

$$3x = 48 \quad (2)$$

$$x = 16 \quad (3)$$

ANALYSIS. We represent the required number by x; and by the condition of the question, $\frac{3}{4}$ of x must be equal to 12, which gives equation (1). Reducing, we find $x = 16$, the answer.

45. The number a is $\frac{3}{4}$ of what number? *Ans.* $\frac{4a}{3}$.

46. The number 21 is $\frac{3}{7}$ of what number? Ans. 49.

47. The number 21 is the $\frac{m}{n}$ part of what number?

Ans. $\frac{21n}{m}$.

48. The number a is the $\frac{m}{n}$ part of what number?

Ans. $\frac{an}{m}$.

49. If you add together $\frac{1}{6}$ and $\frac{1}{7}$ of a certain number, the sum will be 130; what is the number?

SOLUTION.

Let x = the number.

$a = 130$

$$\frac{x}{6} + \frac{x}{7} = a \qquad (1)$$

$$7x + 6x = 42a \qquad (2)$$

$$13x = 42a \qquad (3)$$

$$x = 3\tfrac{3}{13}a \qquad (4)$$

or, $x = 420 \qquad (5)$

ANALYSIS. Representing the required number by x, and the numeral 130 by a, we have by the condition of the problem, equation (1). Reducing, we have $x = 3\frac{3}{13}a$ (4); and restoring the value of a, we have $x = 420$, the number sought.

50. A farmer wishes to mix 116 bushels of provender, consisting of rye, barley, and oats, so that the mixture may contain $\frac{5}{7}$ as much barley as oats, and $\frac{1}{2}$ as much rye as barley; how much of each kind of grain must there be in the mixture?

NOTE.—Instead of employing the literal quantity, a, we may avoid the labor of multiplying numbers together, by *indicating* the operation.

SOLUTION.

Let x = oats;

$\frac{5x}{7}$ = barley;

$\frac{5x}{14}$ = rye.

$$x + \frac{5x}{7} + \frac{5x}{14} = 116 \qquad (1)$$

$$14x + 10x + 5x = 116 \times 14 \qquad (2)$$

$$29x = 116 \times 14 \qquad (3)$$

$$x = 4 \times 14 \qquad (4)$$

or, $x = 56 \qquad (5)$

ANALYSIS. We let x represent the number of bushels of oats. Then, by the conditions, $\frac{5}{7}$ of x, or $\frac{5x}{7}$, must be the barley; and $\frac{1}{2}$ of $\frac{5x}{7}$, or $\frac{5x}{14}$, must be the rye. Putting the sum of all equal to 116, we have equation (1). Multiplying by 14, *indicating* the operation in the second member, we have (2); and reducing we have, finally, $x = 56$.

51. Divide 48 into two such parts, that if the less be divided by 4, and the greater by 6, the sum of the quotients will be 9.

Ans. 12, and 36.

52. A clerk spends $\frac{2}{3}$ of his salary for his board, and $\frac{2}{3}$ of the remainder in clothes, and yet saves \$150 a year. What is his salary? *Ans.* \$1350.

53. An estate is to be divided among 4 children, in the following manner: to the first, \$200 more than $\frac{1}{4}$ of the whole; to the second, \$340 more than $\frac{1}{5}$ of the whole; to the third, \$300 more than $\frac{1}{6}$ of the whole; and to the fourth, \$400 more than $\frac{1}{8}$ of the whole. What is the value of the estate?

Ans. \$4800.

54. Of a detachment of soldiers, $\frac{2}{3}$ are on actual duty, $\frac{1}{8}$ of them are sick, $\frac{1}{5}$ of the remainder absent on leave, and the rest, which is 380, have deserted; what was the number of men in the detachment? *Ans.* 2280 men.

55. A man has a lease for 99 years; being asked how much of it had already expired, he answered that $\frac{2}{3}$ of the time past was equal to $\frac{4}{5}$ of the time to come. Required the time past and the time to come.

Assume $a = 99$. *Ans.* Time past, 54 years.

56. It is required to divide the number 204 into two such parts, that $\frac{2}{5}$ of the less being taken from the greater, the remainder will be equal to $\frac{3}{7}$ of the greater subtracted from 4 times the less. *Ans.* The numbers are 154 and 50.

Put $a = 204$, and restore value in the result.

57. In the composition of a quantity of gunpowder, the nitre was 10 pounds more than $\frac{2}{3}$ of the whole, the sulphur $4\frac{1}{2}$ pounds less than $\frac{1}{6}$ of the whole, and the charcoal 2 pounds less than $\frac{1}{7}$ of the nitre. What was the amount of gunpowder?

SOLUTION.

Let $x=$ the whole;

$\frac{2x}{3}+10=$ the nitre;

$\frac{x}{6}-4\frac{1}{2}=$ the sulphur;

$\frac{2x}{21}+\frac{10}{7}-2=$ the charcoal.

$$\frac{2x}{3}+\frac{x}{6}+\frac{2x}{21}+\frac{10}{7}+3\frac{1}{2} = x. \quad (1)$$
$$4x+x+\frac{4x}{7}+\frac{60}{7}+21 =6x. \quad (2)$$
$$\frac{4x}{7}+\frac{60}{7}+21 = x. \quad (3)$$
$$4x+60+21\times7=7x. \quad (4)$$
$$60+21\times7=3x. \quad (5)$$
$$20+7\times7=x. \quad (6)$$
$$\text{Or,} \quad 69=x. \quad (7)$$

ANALYSIS. Representing the whole composition by x, we obtain expressions for the nitre, sulphur, and charcoal, according to the conditions of the question; and the *sum* of these three ingredients must be equal to x, the whole quantity (1). Multiplying by 6, we have (2); reducing, gives (3); multiplying by 7 we have (4); reducing again, gives (5); dividing by 3, we have (6); and reducing still again, we have $x=69$, the answer.

58. Divide \$44 between three men, A, B, and C, so that the share of A may be $\frac{3}{5}$ that of B, and the share of B $\frac{3}{4}$ that of C. *Ans.* A, \$9; B, \$15; C, \$20.

59. What number is that, to which, if we add its $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, the sum will be 50? *Ans.* 24.

60. What number is that, to which, if we add its $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{4}$, the sum will be a? *Ans.* $\frac{12a}{25}$.

61. In a certain orchard, $\frac{1}{2}$ are apple trees, $\frac{1}{4}$ peach trees, $\frac{1}{6}$ plum trees, 100 cherry trees, and 100 pear trees. How many trees in the orchard? *Ans.* 2400.

62. A farmer has his sheep in five different fields, viz.: $\frac{1}{4}$ in the first field, $\frac{1}{6}$ in the second, $\frac{1}{8}$ in the third, $\frac{1}{12}$ in the fourth, and 45 in the fifth. How many sheep in all the fields? *Ans.* 120

63. After paying out $\frac{1}{4}$ and $\frac{1}{5}$ of my money, I had remaining \$66; how many dollars had I at first? *Ans.* 120.

64. After paying away $\frac{1}{n}$ and $\frac{1}{m}$ of my money, I had a dollars left. How many dollars had I at first?

Ans. $\frac{amn}{mn-m-n}$.

65. If from $\frac{1}{3}$ of my height in inches 12 be subtracted, $\frac{1}{5}$ of the remainder will be 2. What is my height.

Ans. 5 feet 6 inches.

66. A young man, who had just come into possession of a fortune, spent $\frac{3}{8}$ of it the first year, and $\frac{4}{5}$ of the remainder the next year, when he had \$1420 left. What was his fortune.

Ans. \$11360.

67. A person at play lost $\frac{1}{4}$ of his money, and then won 3 shillings; after which he lost $\frac{1}{3}$ of what he then had; and, on counting, found that he had 12 shillings remaining. How much had he at first? *Ans.* 20 shillings.

68. A person at play lost a fourth of his money, and then won 3 shillings; after which he lost a third of what he then had, and then won 2 shillings; lastly, he lost a seventh of what he then had, and then found that he had but 12 shillings remaining. How much had he at first?

Ans. 20 shillings.

69. A shepherd was met by a band of robbers, who plundered him of half of his flock and half a sheep over. Afterward a second party met him, and took half of what he had left and half a sheep over; and soon after this a third party met him and treated him in like manner; and then he had 5 sheep left. How many sheep had he at first? *Ans.* 47.

70. A man bought a horse and chaise for 341(a) dollars. If $\frac{3}{8}$ of the price of the horse be subtracted from twice the

rice of the chaise, the remainder will be the same as if $\frac{4}{7}$ of
ne price of the chaise be subtracted from 3 times the price
f the horse. Required the price of each.

Ans. Horse, \$152; chaise, \$189.

71. If A can build a certain wall in 10 days, and B can do the same in 14 days, what number of days will be required to build the wall, if they both work together?

SOLUTION.

Let $x =$ the number of days;

$\frac{x}{10} =$ A's part of the work;

$\frac{x}{14} =$ B's part of the work.

$$\frac{x}{10} + \frac{x}{14} = 1 \quad (1)$$

$$7x + 5x = 70 \quad (2)$$

$$12x = 70 \quad (3)$$

$$x = 5\tfrac{5}{6} \quad (4)$$

ANALYSIS. Let x represent the number of days both are to work. Since A will build $\frac{1}{10}$ of the wall in 1 day, he will build $\frac{x}{10}$ of it in x days; and since B will build $\frac{1}{14}$ in 1 day, he will build $\frac{x}{14}$ in x days. But since the wall is to be *completed* by both working x days, the sum of these two fractions must be equal to unity; hence equation (1), which, reduced, gives $5\frac{5}{6}$, the number of days required.

72. If A can do a piece of work in a days, and B can do the same in b days, how long will it take them if they both work together?

Ans. $\frac{ab}{a+b}$ days.

73. A can do a piece of work in 12 days, and B can do the same in 24 days; how many days will be required, if they both work together? *Ans.* 8.

74. A laborer, A, can perform a piece of work in 5 days, B can do the same in 6 days, and C in 8 days; in what time can the three together perform the same work?

Ans. $2\frac{2}{59}$ days.

75. A laborer engaged to serve for 60 days on these conditions: for every day he worked he should have 75 cents and his board, and for every day he was idle he should forfeit 25 cents for damage and board. At the end of the time he received \$25. How many days did he work, and how many days was he idle?

SOLUTION.

Let $x =$ days he was idle;
$60 - x =$ days he worked.

$$45 - x = 25 \qquad (1)$$
$$x = 20 \qquad (2)$$
$$60 - x = 40$$

ANALYSIS. Representing the idle days by x, $60 - x$ must be the working days. Had he worked the whole 60 days, his wages would have amounted to \$45. But for every day he was idle, he not only lost his wages, 75 cents, but 25 cents in addition, making \$1 a day. In x idle days, therefore, he lost x dollars. Consequently, the amount due him was $45 - x$ dollars. But by the conditions of the problem, the money due him was 25 dollars; hence we have (1), from which we find he was idle 20 days, and worked 40 days.

76. A person engaged to work a days on these conditions; for each day he worked he was to receive b cents, and for each day he was idle he was to forfeit c cents. At the end of a days he received d cents; how many days was he idle?

Ans. $\dfrac{ab - d}{b + c}$.

77. A boy engaged to convey 30 glass vessels to a certain place, on condition of receiving 5 cents for every one he delivered safe, and forfeiting 12 cents for every one he broke. On settlement, he received 99 cents; how many did he break?

Ans. 3.

78. A boy engaged to carry n glass vessels to a certain place on these conditions: he was to receive a cents for every one he delivered safe, and to forfeit b cents for every one he broke. On settlement he received d cents; how many did he break?

Ans. The number represented by $\dfrac{an - d}{a + b}$

SIMPLE EQUATIONS

CONTAINING TWO UNKNOWN QUANTITIES.

146. Independent Equations are such as cannot be reduced to the same form, or derived one from the other; as $x + 3y = a$, and $4x + 5y = b$. Independent equations refer to the same problem, and express different conditions of the problem.

147. We have seen that, in order to find the value of any unknown quantity in an equation, we separate it from the other quantities, and cause it to stand alone as one member of the equation. But if the equation contain two unknown quantities, the value of neither can be determined by this process. To show the reason of this, let us consider the following equation:

$$x + y = 20. \quad (1)$$

Transposing y, we have

$$x = 20 - y,$$

in which x is still undetermined, because its value in the second member of the equation contains the unknown quantity, y.

Again, transposing x in equation (1), we have

$$y = 20 - x,$$

in which y is still unknown, because its value contains the unknown quantity, x. Hence,

Two unknown quantities cannot be determined from a single equation.

The equation given above expresses this condition: viz., *the sum of two numbers is* 20; and since there are many pairs or couplets of numbers of which the sum is 20, x and y can have no particular or exclusive values. The equation is satis-

---

Define independent equations. They always refer to what? Why cannot the values of two unknown quantities be determined from one equation?

fied if we make $x = 1$ and $y = 19$; or, $x = 2$ and $y = 18$, etc.; for

$$1 + 19 = 20,\ 2 + 18 = 20, \text{ etc.}$$

But if we combine another equation with this, as $x - y = 4$, which expresses a different condition: viz., *that the difference of the two numbers is* 4, then only one value for x and one value for y will satisfy both equations, or answer both conditions. To find these values we may proceed thus:

$$x + y = 20 \quad (1)$$
$$x - y = 4 \quad (2)$$

By addition, $2x = 24$, or $x = 12$

Subtracting (2) from (1), $2y = 16$, or $y = 8$

And 8 and 12 are the *only* numbers whose sum is 20 and difference 4. From this result we learn that

Two unknown quantities can be determined from two independent equations.

To effect the reduction, we must derive from the two a new equation containing but one unknown quantity. This operation is called

ELIMINATION.

148. **Elimination** is the process of combining two or more equations, containing two or more unknown quantities, in such a manner as to cause one or more of the unknown quantities contained in them to disappear.

There are three principal methods of elimination:

1st, *By substitution;* 2d, *By comparison;* 3d, *By addition and subtraction.*

CASE I.

149. Elimination by substitution.

1. Given $2x + 5y = 31$, and $3x + 2y = 19$, to find the values of x and y.

How many equations are required that the values of two unknown *quantities* may be determined? Why? Define elimination. Name the *methods.* Give Case I.

OPERATION.

$$2x + 5y = 31 \qquad \text{(A)}$$
$$3x + 2y = 19 \qquad \text{(B)}$$

$$x = \frac{31 - 5y}{2} \qquad (1)$$
$$\frac{93 - 15y}{2} + 2y = 19 \qquad (2)$$
$$93 - 15y + 4y = 38 \qquad (3)$$
$$-11y = -55 \qquad (4)$$
$$y = 5 \qquad (5)$$
$$x = \frac{31 - 25}{2} \qquad (6)$$
$$x = 3 \qquad (7)$$

ANALYSIS. We transpose $5y$ in equation (A), and divide by 2, and obtain (1), which expresses the algebraic value of x. This *value* of x we substitute *for* x in (B); thus, instead of $3x$, we write its value, 3 times $\frac{31-5y}{2}$, or $\frac{93-15y}{2}$; which with the other terms of equation (B) written in their order, gives (2), an equation containing only one unknown quantity, y; therefore x has been eliminated. Reducing in the usual manner, we have $y = 5$. Since y is 5, $5y$ is 25; and substituting this value in the second member of equation (1), we have (6), which gives the value of x in known terms. Reducing, we obtain $x = 3$. Hence, the following

RULE. I. *Find the value of one of the unknown quantities in one of the given equations.*

II. *Substitute this value for the same unknown quantity in the other equation.*

EXAMPLES FOR PRACTICE.

2. Given $\left\{\begin{array}{l} 3x + 2y = 23 \\ x + 4y = 21 \end{array}\right\}$ to find x and y.

Ans. $x = 5$; $y = 4$.

3. Given $\left\{\begin{array}{l} 4x - y = 5 \\ 2x + 2y = 20 \end{array}\right\}$ to find x and y.

Ans. $x = 3$; $y = 7$.

4. Given $\left\{\begin{array}{l} x + 3y = 10 \\ x - y = 6 \end{array}\right\}$ to find x and y.

Ans. $x = 7$; $y = 1$.

Give Analysis. Rule.

5 Given $\left\{\begin{array}{l} 2y + 4x = 62 \\ y + 5x = 73 \end{array}\right\}$ to find y and x.

Ans. $y = 3$; $x = 14$.

6. Given $\left\{\begin{array}{l} 2y + 5x = 29 \\ 2y - x = -1 \end{array}\right\}$ to find y and x.

Ans. $y = 2$; $x = 5$.

7. Given $\left\{\begin{array}{l} 3x + 4z = 37 \\ x + 5z = 27 \end{array}\right\}$ to find x and z.

Ans. $x = 7$; $z = 4$.

8. Given $\left\{\begin{array}{l} y - z = 4 \\ 4y - 2z = 36 \end{array}\right\}$ to find y and z.

Ans. $y = 14$; $z = 10$.

CASE II.

150. Elimination by comparison.

1. Given $3x + 2y = 16$, and $4x + 3y = 23$, to find the values of x and y.

OPERATION.

$$3x + 2y = 16 \quad \text{(A)}$$
$$4x + 3y = 23 \quad \text{(B)}$$
$$x = \frac{16 - 2y}{3} \quad (1)$$
$$x = \frac{23 - 3y}{4} \quad (2)$$
$$\frac{16 - 2y}{3} = \frac{23 - 3y}{4} \quad (3)$$
$$64 - 8y = 69 - 9y \quad (4)$$
$$y = 5 \quad (5)$$
$$3x + 10 = 16 \quad (6)$$
$$x = 2 \quad (7)$$

ANALYSIS. Transposing $2y$ in (A), and dividing by 3, we obtain (1). From (B), in like manner, we obtain (2). We have thus found two algebraic values for x which must be equal to each other. (Ax. 7.). Equating these values, we have (3), an equation containing but one unknown quantity, y; consequently, x has been eliminated. Reducing this equation, we obtain $y=5$. Since y is 5, $2y$ is 10; and substituting this value in (A), we have (6) which reduced, gives $x=2$.

Give Case II. Analysis.

Hence, the following

RULE. I. *Find the value of the same unknown quantity n each of the given equations.*

II. *Form an equation by placing these values equal to each other.*

EXAMPLES FOR PRACTICE.

2. Given $\begin{cases} 3x + 5y = 29 \\ 4x - 2y = 4 \end{cases}$ to find x and y.

Ans. $x = 3$; $y = 4$.

3. Given $\begin{cases} y - x = 4 \\ y + 4x = 9 \end{cases}$ to find y and x.

Ans. $y = 5$; $x = 1$.

4. Given $\begin{cases} 2x + 5z = 29 \\ x - 2z = 1 \end{cases}$ to find x and z.

Ans. $x = 7$; $z = 3$.

5. Given $\begin{cases} z + 11y = 134 \\ 2z - y = 15 \end{cases}$ to find z and y.

Ans. $z = 13$; $y = 11$.

6. Given $\begin{cases} 12x + 14y = 26 \\ 3x - y = 2 \end{cases}$ to find x and y.

Ans. $x = 1$; $y = 1$.

7. Given $\begin{cases} 4x + 12y = 5 \\ x + 2y = 1 \end{cases}$ to find x and y.

Ans. $x = \frac{1}{2}$; $y = \frac{1}{4}$.

8. Given $\begin{cases} \dfrac{x}{4} + \dfrac{5y}{6} = 14 \\ \dfrac{x}{8} + \dfrac{y}{6} = 4 \end{cases}$ to find x and y.

Ans. $x = 16$; $y = 12$.

9. Given $\begin{cases} \dfrac{4x + 3y}{9} + y = 4 \\ x + \dfrac{8x - 2y}{5} = 7 \end{cases}$ to find x and y.

Ans. $x = 3$; $y = 2$.

Give Rule.

CASE III.

151. Elimination by addition or subtraction.

1. Given $6x + 9y = 48$, and $4x + 2y = 16$, to find the values of x and y.

OPERATION.

$$
\begin{array}{rlr}
6x + 9y &= 48 & \text{(A)} \\
4x + 2y &= 16 & \text{(B)} \\
\hline
12x + 18y &= 96 & (1) \\
12x + 6y &= 48 & (2) \\
\hline
12y &= 48 & (3) \\
y &= 4 & (4) \\
\hline
4x + 8 &= 16 & (5) \\
4x &= 8 & (6) \\
x &= 2 & (7)
\end{array}
$$

ANALYSIS. We multiply equation (A) by 2, and obtain (1); we next multiply equation (B) by 3, and obtain (2). We have thus obtained two equations in which the coefficients of x are equal. Subtracting equation (2) from equation (1), member from member, the terms containing x cancel, and we have (3), an equation containing but one unknown quantity, y. Reducing, we have $y = 4$. Substituting this value of y in (B) we have (5), which reduced gives $x = 2$.

2. Given $7x - 2y = 31$, and $5x + 2y = 29$, to find x and y.

OPERATION.

$$
\begin{array}{rlr}
7x - 2y &= 31 & \text{(A)} \\
5x + 2y &= 29 & \text{(B)} \\
\hline
12x &= 60 & (1) \\
x &= 5 & (2) \\
\hline
25 + 2y &= 29 & (3) \\
2y &= 4 & (4) \\
y &= 2 & (5)
\end{array}
$$

ANALYSIS. Since the coefficients of y, in the given equations, are numerically equal, and have opposite signs, their sum is zero, and y will be eliminated by *adding* equations (A) and (B), member by member; this gives equation (1), from which we obtain $x = 5$. Substituting this value of x in (B), we obtain (3), which gives $y = 2$.

From these examples we derive the following

RULE. I. *Multiply or divide the equations by such numbers or quantities that the coefficients of the quantity to be eliminated shall be made equal.*

Give Case III. Analyses. Rule.

I. *If these coefficients have unlike signs, add the prepared quations; if like signs, subtract one equation from the her.*

NOTES. 1. If the given equations require to be multiplied, find the ast common multiple of the coefficients of the quantity to be eliminated, d divide it by each coefficient; the quotients will be the least multiiers that can be used.

2. If the coefficients are prime to each other, multiply each equation y the coefficient in the other equation.

3. If necessary, clear the equations of fractions.

EXAMPLES FOR PRACTICE.

3. Given $\left\{\begin{array}{l} 2x + 4y = 28 \\ 4x + 3y = 31 \end{array}\right\}$ to find x and y.

Ans. $x = 4$; $y = 5$.

4. Given $\left\{\begin{array}{l} 5x - 2y = 8 \\ 7x - y = 13 \end{array}\right\}$ to find x and y.

Ans. $x = 2$; $y = 1$.

5. Given $\left\{\begin{array}{l} 4x + y = 13 \\ 6x + 12y \quad 72 \end{array}\right\}$ to find x and y.

Ans. $x = 2$; $y = 5$.

6. Given $\left\{\begin{array}{l} 3x + z = 41 \\ 7x - z = 79 \end{array}\right\}$ to find x and z.

Ans. $x = 12$; $z = 5$.

7. Given $\left\{\begin{array}{l} 2x + 2y = 26 \\ 7x - y = 67 \end{array}\right\}$ to find x and y.

Ans. $x = 10$; $y = 3$.

8. Given $\left\{\begin{array}{l} 3x + 5y = 2 \\ 15x - 5y = 4 \end{array}\right\}$ to find x and y.

Ans. $x = \frac{1}{3}$; $y = \frac{1}{5}$.

9. Given $\left\{\begin{array}{l} \frac{3x}{7} + \frac{2y}{3} = 5 \\ x - y = 4 \end{array}\right\}$ to find x and y.

Ans. $x = 7$; $y = 3$.

10. Given $\left\{\begin{array}{l} \frac{x-4}{3} - \frac{y-3}{3} = 0 \\ 2x - \frac{3y-6}{4} = 21 \end{array}\right\}$ to find x and y.

Ans. $x = 12$; $y =$

152. Of these three methods of elimination, sometimes one is preferable and sometimes another, according to the relation of the coefficients and the positions in which they stand.

No one should be prejudiced against either method; and in practice we use either one, or modifications of either, as the case may require. This precept may be applied in the solution of the following

GENERAL EXAMPLES.

1. Given $\left\{\begin{array}{r} 4x + y = 34 \\ x + 4y = 16 \end{array}\right\}$ to find x and y.

Ans. $y = 2$; $x = 8$.

2. Given $\left\{\begin{array}{r} 7x + 4y = 58 \\ 9x - 4y = 38 \end{array}\right\}$ to find x and y.

Ans. $x = 6$; $y = 4$.

3. Given $\left\{\begin{array}{r} 5x + 6y = 58 \\ 2x + 6y = 34 \end{array}\right\}$ to find x and y.

Ans. $x = 8$; $y = 3$.

4. Given $\left\{\begin{array}{r} 4x + 3y = 22 \\ -4x + 2y = -12 \end{array}\right\}$ to find x and y.

Ans. $x = 4$; $y = 2$.

5. Given $\left\{\begin{array}{r} 6x + 5y = 128 \\ 3x + 4y = 88 \end{array}\right\}$ to find x and y.

Ans. $x = 8$; $y = 16$.

6. Given $\left\{\begin{array}{r} 2x + 3z = 38 \\ 6x + 5z = 82 \end{array}\right\}$ to find x and z.

Ans. $x = 7$; $z = 8$.

7. Given $\left\{\begin{array}{r} 4x + 6y = 46 \\ 5x - 2y = 10 \end{array}\right\}$ to find x and y.

Ans. $x = 4$; $y = 5$.

8 Given $\left\{\begin{array}{r} 2x + 3y = 31 \\ 4x - 3y = 17 \end{array}\right\}$ to find x and y.

Ans. $x = 8$; $y = 5$.

9. Given $\left\{\begin{array}{r} 4y + z = 102 \\ y + 4z = 48 \end{array}\right\}$ to find y and z.

Ans. $y = 24$; $z = 6$.

10 Given $\left\{\begin{array}{l} 2x + 3y = 7 \\ 8x + 10y = 26 \end{array}\right\}$ to find x and y.

Ans. $x = 2$; $y = 1$.

11. Given $\left\{\begin{array}{l} 5y + 3x = 93 \\ 3y + 4x = 80 \end{array}\right\}$ to find y and x.

Ans. $y = 12$; $x = 11$.

12. Given $\left\{\begin{array}{l} \frac{1}{2}x + \frac{1}{3}y = 14 \\ \frac{1}{3}x + \frac{1}{2}y = 11 \end{array}\right\}$ to find x and y.

Ans. $x = 24$; $y = 6$.

13. Given $\left\{\begin{array}{l} x + \frac{1}{2}y = 8 \\ \frac{1}{2}x + y = 7 \end{array}\right\}$ to find x and y.

Ans. $x = 6$; $y = 4$.

14. Given $\left\{\begin{array}{l} \frac{1}{7}x + 7y = 99 \\ \frac{1}{7}y + 7x = 51 \end{array}\right\}$ to find x and y.

Ans. $x = 7$; $y = 14$.

15. Given $\left\{\begin{array}{l} \dfrac{x-y}{5} + \dfrac{x+y}{19} = 4 \\ x - y = 10 \end{array}\right\}$ to find x and y.

Ans. $x = 24$; $y = 14$.

16. Given $\left\{\begin{array}{l} x + y = 2a \\ x - y = 2b \end{array}\right\}$ to find x and y.

Ans. $x = a + b$; $y = a - b$.

17. Given $\left\{\begin{array}{l} 4x - 2y = 8d - 2a \\ x + y = a + 3c + 2d \end{array}\right\}$ to find x and y.

Ans. $x = c + 2d$; $y = a + 2c$.

18. Given $\left\{\begin{array}{l} cx + my = 2a \\ cx - my = 2b \end{array}\right\}$ to find x and y.

Ans. $x = \dfrac{a+b}{c}$; $y = \dfrac{a-b}{m}$.

19. Given $\left\{\begin{array}{l} ax + cy = b \\ ax - by = c \end{array}\right\}$ to find x and y.

Ans. $x = \dfrac{b^2 + c^2}{a(b+c)}$; $y = \dfrac{b-c}{b+c}$.

20. Given $\left\{\begin{array}{l} \dfrac{x}{m} + ny = m + n \\ \dfrac{mx}{n} + \dfrac{n^3 y}{m} = m^2 + n^2 \end{array}\right\}$ to find x and y.

Ans. $x = mn$; $y = \dfrac{m}{n}$.

PROBLEMS

PRODUCING EQUATIONS CONTAINING TWO UNKNOWN QUANTITIES

153. Many of the problems hitherto given require the determination of more than one unknown quantity; but the quantities are so related to each other that they can be expressed algebraically by the use of a single letter. The solution is often rendered more simple by using as many letters as there are quantities to be determined. If two letters are employed to represent unknown quantities, the conditions of the problem must furnish two independent equations; otherwise it will not be capable of solution (**147**).

1. A man bought at one time 3 bushels of wheat and five bushels of rye for 38 shillings; and at another time, 6 bushels of wheat and 3 bushels of rye for 48 shillings. What was the price of a bushel of each?

SOLUTION.

Let $x =$ price of wheat;
$y =$ price of rye.

$$
\begin{array}{rll}
3x + 5y = 38 & \text{(A)} \\
6x + 3y = 48 & \text{(B)} \\
6x + 10y = 76 & (1) \\
6x + 3y = 48 & (2) \\
7y = 28 & (3) \\
y = 4 & (4) \\
3x + 20 = 38 & (5) \\
x = 6 & (6)
\end{array}
$$

ANALYSIS. We represent the prices by x and y. Since 3 bushels of wheat and 5 bushels of rye cost 38 shillings, we have equation (A), furnished by the first condition; and since 6 bushels of wheat and 3 bushels of rye cost 48 shillings, we have equation (B), from the second condition. Multiplying (A) by (2) to make the coefficients of x equal, equations (A) and (B) become (1) and (2). Subtracting (2) from (1), we have (3), which, reduced, gives $y = 4$. Substituting this value of y in (A), we have (5), which gives $y = 6$.

2. A man spent 30 cents for apples and pears, buying his apples at the rate of 4 for a cent, and his pears at the rate of 5 for a cent. He afterward let a friend have half of his apples and one-third of his pears, for 13 cents, at the same rate. How many did he buy of each sort?

SOLUTION.

Let $x =$ number of apples, and $y =$ number of pears.

Hence, $\frac{x}{4} =$ cost of apples, and $\frac{y}{5} =$ cost of pears.

By the first condition, $\frac{x}{4} + \frac{y}{5} = 30$ (A)

By the second condition, $\frac{x}{8} + \frac{y}{15} = 13$ (B)

Multiplying (B) by 2, $\frac{x}{4} + \frac{2y}{15} = 26$ (1)

Subtracting (1) from (A) $\frac{y}{5} - \frac{2y}{15} = 4$ (2)

Reducing, $y = 60$ (3)

Substituting in (A), $\frac{x}{4} + 12 = 30$ (4)

Reducing, $x = 72$ (5)

3. What fraction is that, to the numerator of which, if 1 be dded, its value will be $\frac{1}{3}$, but if 1 be added to the denominaor, its value will be $\frac{1}{4}$?

SOLUTION.

Let $\frac{x}{y} =$ the fraction.

By the first condition, $\frac{x+1}{y} = \frac{1}{3}$ (A)

By the second condition, $\frac{x}{y+1} = \frac{1}{4}$ (B)

Clearing (A) of fractions, $3x + 3 = y$ (1)

Clearing (B) of fractions, $4x - 1 = y$ (2)

Subtracting (1) from (2), $x - 4 = 0$ (3)

Transposing, $x = 4$ (4)

From (1) $y = 15$ (5)

Hence, the fraction is $\frac{x}{y} = \frac{4}{15}$ (6)

19. What two numbers are those, whose sum is a and difference b?

SOLUTION.

Let $x =$ the greater.
And $y =$ the less.

From the first condition, $x + y = a$ (1)
From the second condition, $x - y = b$ (2)

Adding (2) to (1), $2x = a + b$ (3)
Subtracting (2) from (1), $2y = a - b$ (4)

Hence, $x = \frac{a}{2} + \frac{b}{2}$, greater.

And $y = \frac{a}{2} - \frac{b}{2}$, less.

In these results, we have the algebraic expression of the following general truth:

The half sum of any two numbers, added to the half difference, is the greater of the two numbers; and the half sum, diminished by the half difference, is the less

20. The sum of two numbers is 28, and their difference is 6; what are the numbers?

Ans. Greater, 17; less, 11.

21. There are two numbers whose sum is 100, and three times the less taken from twice the greater, gives 150 for remainder. What are the numbers?

Ans. 90 and 10.

22. A man and his wife labored m days, and received $2a$ dollars for compensation. Had the wife been idle, and on expense at the same daily rate as her wages, they would have saved but $2c$ dollars. What were the daily wages of each?

Ans. Man's, $\frac{a+c}{m}$; wife's, $\frac{a-c}{m}$.

Having the sum and the difference of two numbers given, how may the numbers be found?

23. Find two numbers, such that twice the first diminished by the second shall be equal to $3b$; and twice the second diminished by the first shall be equal to $3a$.

Ans. First, $a+2b$; second, $2a+b$.

24. Divide the number a into two such parts that the first shall be to the second as m to n.

Ans. First, $\frac{ma}{m+n}$; second, $\frac{na}{m+n}$.

SIMPLE EQUATIONS

CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

154. 1. Given $\left\{\begin{array}{l} x+y+z=9 \\ x+2y+3z=16 \\ x+3y+4z=21 \end{array}\right\}$ to find x, y, and z.

OPERATION.

$$\begin{array}{rl}
x+y+z=9 & \text{(A)} \\
x+2y+3z=16 & \text{(B)} \\
x+3y+4z=21 & \text{(C)} \\
\hline
x=9-y-z & (1) \\
x=16-2y-3z & (2) \\
x=21-3y-4z & (3) \\
\hline
9-y-z=16-2y-3z & (4) \\
16-2y-3z=21-3y-4z & (5) \\
y=7-2z & (6) \\
y=5-z & (7) \\
\hline
5-z=7-2z & (8) \\
\text{Ans.}\left\{\begin{array}{l} z=2 \\ y=3 \\ x=4 \end{array}\right. & \begin{array}{l} (9) \\ (10) \\ (11) \end{array}
\end{array}$$

ANALYSIS. By transposing, we obtain equation (1) from (A), (2) from (B), and (3) from (C). These equations give us three values for x. Equating the 1st and 2d values, we have (4), and equating the 2d and 3d values, we have (5). We have thus eliminated x, and obtained two equations with two unknown quantities. By transposing terms in (4) and (5) and reducing, we have (6) and (7), giving two values for y. Equating these values, we eliminate y, and obtain (8), from which we find $z=2$. By substi-

tuting the value of z in (7) we obtain $y = 3$; and substituting the values of z and y in (1) we obtain $x = 4$.

In this example we have eliminated by the method of *comparison.*

2. Given $\left\{\begin{array}{l} 2x + 4y - 3z = 22 \\ 4x - 2y + 5z = 18 \\ 6x + 7y - z = 63 \end{array}\right\}$ to find x, y and z.

OPERATION.

$$\begin{array}{rll}
2x + 4y - 3z = 22 & \text{(A)} \\
4x - 2y + 5z = 18 & \text{(B)} \\
6x + 7y - z = 63 & \text{(C)} \\ \hline
4x + 8y - 6z = 44 & \text{(1)} \\
4x - 2y + 5z = 18 & \\ \hline
10y - 11z = 26 & \text{(2)} \\ \hline
6x + 12y - 9z = 66 & \text{(3)} \\
6x + 7y - z = 63 & \\ \hline
5y - 8z = 3 & \text{(4)} \\ \hline
10y - 16z = 6 & \text{(5)} \\
10y - 11z = 26 & \\ \hline
5z = 20 & \text{(6)} \\
z = 4 & \text{(7)} \\
y = 7 & \text{(8)} \\
x = 3 & \text{(9)}
\end{array}$$

ANALYSIS. We multiply (A) by 2, and obtain (1). Writing (B) underneath this and subtracting, we eliminate x, and obtain (2). Next, multiplying (A) by 3, we obtain (3). Writing (C) under this and subtracting, we again eliminate x, and obtain (4). Multiplying (4) by 2 gives (5). Writing (2) under this and subtracting (5) from it, we eliminate y and obtain (6), which reduced, gives $z = 4$. Substituting this value of z in (4), and reducing, we have $y = 7$; and substituting the values of z and y in (A), and reducing, we find $x = 3$.

In this example we have eliminated principally by the method of *addition or subtraction.*

From the illustrations given we deduce the following

RULE. I. *Combine one of the equations with each of the others, eliminating successively the same unknown quantity: the result will be a new set of equations containing one less unknown quantity.*

II. *Combine one of the new equations with each of the others, eliminating a second unknown quantity; and the re-*

Give rule for reducing equations containing three or more unknown quantities.

sult will be a new set containing two unknown quantities less than the original.

III. *Continue this process till an equation is found containing but one unknown quantity.*

IV. *Reduce this equation and find the value of the unknown quantity. Substitute this value in an equation containing two unknown quantities, and thus find the value of a second. Substitute these values in an equation containing three unknown quantities, and find the value of a third; and so on, till the values of all are found.*

NOTE.—Instead of combining the first with each of the others, we may combine the first with the second, the second with the third, and so on; or we may pursue any other order of combination best suited to the mutual relations of the coefficients.

155. It is evident that if there are more unknown quantities than equations, the last resulting equation will contain two or more unknown quantities, and the solution will be impossible (**147**). Hence the following general law:

There must be as many independent equations as there are unknown quantities.

NOTE.—If there are more independent equations than unknown quantities, some of them can be dispensed with in reducing the equations.

EXAMPLES FOR PRACTICE.

1. Given $\left\{\begin{array}{r} 3x + 9y + 8z = 41 \\ 5x + 4y - 2z = 20 \\ 11x + 7y - 6z = 37 \end{array}\right\}$ to find x, y, and z.

Ans. $x = 2$; $y = 3$; $z = 1$

2. Given $\left\{\begin{array}{r} 3x + 5y + z = 26 \\ 6x + 3y + 2z = 31 \\ 9x + 4y + 4z = 50 \end{array}\right\}$ to find x, y, and z.

Ans. $x = 2$; $y = 3$; $z = 5$.

What number of equations is required in the solution of a problem? Why?

NOTE —When several coefficients are unity, or multiples of each other, certain *expedients* may be employed to facilitate the calculation, for which no specific rules can be given.

3. Given $\left\{\begin{array}{l} x+y+z=31 \\ x+y-z=25 \\ x-y-z=\ \ 9 \end{array}\right\}$ to find x, y, and z.

Subtract the 2d from the 1st, and $2z=6$, or $z=3$.
Subtract the 3d from the 2d, and $2y=16$, or $y=8$.
Add the 1st and 3d, and $2x=40$, or $x=20$.

4. Given $\left\{\begin{array}{l} u+v+x+y=10 \\ u+v+z+x=11 \\ u+v+z+y=12 \\ u+x+y+z=13 \\ v+x+y+z=14 \end{array}\right\}$ to find the value of each.

Since in each equation one letter is wanting, assume

$$u+v+x+y+z=s$$

Then

$$\begin{aligned} s-z&=10 \\ s-y&=11 \\ s-x&=12 \\ s-v&=13 \\ s-u&=14 \end{aligned}$$

By addition,

$$\begin{aligned} 5s-s&=60 \\ s&=15 \end{aligned}$$

Hence, by substituting the value of s, $\left\{\begin{array}{l} z=5 \\ y=4 \\ x=3 \\ v=2 \\ u=1 \end{array}\right.$

Required, the values of the unknown quantities in the following equations:

5. $\left\{\begin{array}{l} x+y+z=26 \\ x-y\qquad\ =\ 4 \\ x-z\qquad\ =\ 6 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=12, \\ y=\ 8, \\ z=\ 6. \end{array}\right.$

6 $\left\{\begin{array}{l} x-y-z=\ 6 \\ 3y-x-z=12 \\ 7z-y-x=24 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=39, \\ y=21, \\ z=12. \end{array}\right.$

7. $\left\{\begin{array}{l} x+\frac{1}{2}y=100 \\ y+\frac{1}{3}z=100 \\ z+\frac{1}{4}x=100 \end{array}\right\}$ *Ans.* $\left\{\begin{array}{l} x=64, \\ y=72, \\ z=84. \end{array}\right.$

8. $\begin{cases} x + y = 52 \\ y + z = 82 \\ z + w = 68 \\ w + u = 30 \\ u + x = 32 \end{cases}$ *Ans.* $\begin{cases} x = 20, \\ y = 32, \\ z = 50, \\ w = 18, \\ u = 12. \end{cases}$

9. $\begin{cases} \frac{1}{3}x + 3y = 23 \\ x + \frac{z}{4} = 8 \\ y + 3z = 31 \\ x + y + z + 2w = 39 \end{cases}$ *Ans.* $\begin{cases} x = 6, \\ y = 7, \\ z = 8, \\ w = 9. \end{cases}$

10. $\begin{cases} 4x + 2y - 3z = 4 \\ 3x - 5y + 2z = 22 \\ x + y + z = 12 \end{cases}$ *Ans.* $\begin{cases} x = 5, \\ y = 1, \\ z = 6. \end{cases}$

11. $\begin{cases} \frac{1}{2}x + \frac{1}{3}y + \frac{1}{4}z = 22 \\ \frac{1}{4}x + y + \frac{1}{2}z = 33 \\ x + \frac{1}{4}y - \frac{1}{8}z = 19 \end{cases}$ *Ans.* $\begin{cases} x = 20, \\ y = 12, \\ z = 32. \end{cases}$

12. $\begin{cases} x + y = a \\ x + z = b \\ y + z = c \end{cases}$ *Ans.* $\begin{cases} x = \frac{1}{2}(a + b - c), \\ y = \frac{1}{2}(a + c - b), \\ z = \frac{1}{2}(b + c - a). \end{cases}$

13. $\begin{cases} cx + y + az = a + ac + c \\ c^2x + y + a^2z = 3ac \\ acx + 2y + acz = a^2 + 2ac + c^2 \end{cases}$ *Ans.* $\begin{cases} x = \frac{a}{c}, \\ y = ac, \\ z = \frac{c}{a}. \end{cases}$

PROBLEMS

PRODUCING EQUATIONS CONTAINING THREE OR MORE UNKNOWN QUANTITIES.

156. 1. Three persons, A, B, and C, talking of their ages, it was discovered that the sum of their ages was 90; the half sum of A's and B's was 25; and the half sum of B's and C's was 35. What was the age of each?

Ans. A's = 20 years; B's = 30; and C's = 40.

2. A's money added to 3 times B's and C's will amount to \$470; B's money added to 4 times A's and C's will amount to \$580; and C's money added to 5 times A's and B's, will amount to \$630. How much money has each?

SOLUTION.

By the conditions,

$$\begin{cases} x + 3y + 3z = 470. & (1) \\ y + 4x + 4z = 580. & (2) \\ z + 5x + 5y = 630. & (3) \end{cases}$$

Adding $2x$ to (1),
" $3y$ to (2),
" $4z$ to (3),

$$\begin{cases} 3x + 3y + 3z = 470 + 2x & (4) \\ 4x + 4y + 4z = 580 + 3y & (5) \\ 5x + 5y + 5z = 630 + 4z & (6) \end{cases}$$

Assume

$$s = x + y + z.$$

Equation (4) becomes
" (5) "
" (6) "

$$\begin{cases} 3s = 470 + 2x & (7) \\ 4s = 580 + 3y & (8) \\ 5s = 630 + 4z & (9) \end{cases}$$

Multiplying (7) by 6,
" (8) by 4,
" (9) by 3,

$$\begin{cases} 18s = 2820 + 12x & (10) \\ 16s = 2320 + 12y & (11) \\ 15s = 1890 + 12z & (12) \end{cases}$$

Adding (10), (11), and (12),

$$49s = 7030 + 12s \quad (13)$$
$$37s = 7030 \quad (14)$$
$$s = 190 \quad (15)$$

Substituting value of s in (7),
" " " " (8),
" " " " (9),

$$\begin{cases} x = \$50, & \text{A's}, \\ y = 60, & \text{B's}, \\ z = 80, & \text{C's}. \end{cases}$$

NOTE.—The pupil, if he choose, may solve the above question by the usual methods. But the solution given, to which we call particular attention, is calculated to impart a superior skill, and to cultivate a higher mathematical taste. Similar expedients may be employed in several of the following problems.

3. A farmer has sheep in three pastures. The number in the first pasture, added to half the number in the second and third, will make 70. The number in the second pasture, added

to one third of the number in the first and third, will make 60. And the number in the third pasture, added to one fifth of the number in the other two, will make 58. How many sheep in each pasture? *Ans.* In the first, 30; second, 35; third, 45.

4. Three persons divided a sum of money among them in such a manner that the shares of A and B together amounted to $900, the shares of A and C together to $800, and the shares of B and C to $700; what was the share of each?

Ans. A's share, $500; B's, $400; and C's, $300.

5. The sum of three numbers is 59; one half the difference of the first and second is 5, and one half the difference of the first and third is 9; required the numbers.

Ans. 29, 19, and 11.

6. A certain number, consisting of two places, a unit and a ten, is four times the sum of its digits, and if 27 be added to it, the digits will be inverted. What is the number?

NOTE.—The local value of a figure is increased tenfold by every remove to the left of the unit's place; hence if x represent a digit in the place of tens, and y in the place of units, the number will be expressed by $10x + y$. A number consisting of three places, with x, y, and z, to represent the digits, will be expressed by $100x + 10y + z$.

Ans. 36.

7. A number is expressed by three figures; the sum of these figures is 9; the figure in the place of units is double that in the place of hundreds, and when 198 is added to this number, the sum obtained is expressed by the figures of this number reversed; what is the number? *Ans.* 234.

8. Divide the number 90 into three parts, such that twice the first part increased by 40, three times the second part increased by 20, and four times the third part increased by 10, may be all equal to one another.

Ans. First part, 35; second, 30; and third, 25.

9. Find three numbers, such that the first with $\frac{1}{3}$ of the other two, the second with $\frac{1}{4}$ of the other two, or the third with $\frac{1}{5}$ of the other two, shall be equal to 25.

Ans. 13, 17, and 19.

10. There are three numbers, such that the first with $\frac{1}{2}$ the second, is equal to 14; the second with $\frac{1}{3}$ of the third, is equal to 18; and the third with $\frac{1}{4}$ of the first, is equal to 20; required the numbers. *Ans.* 8, 12, and 18.

11. Find three members, such that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third shall be equal to 62; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third equal to 47; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third equal to 38.

Ans. 24, 60, and 120.

12. Find three numbers of such magnitude that the first with the $\frac{1}{2}$ sum of the other two, the second with $\frac{1}{3}$ of the other two, and the third with $\frac{1}{4}$ of the other two, may be the same, and amount to 51 in each case. *Ans.* 15, 33, and 39.

13. Four boys, A, B, C, and D, comparing their money, found that A's money added to $\frac{1}{3}$ of the sum possessed by the other three, would make \$30; B's money added to $\frac{1}{3}$ of the sum possessed by the other three, would make \$32; C's money added to $\frac{1}{3}$ of the sum possessed by the other three, would make \$34; and D's money added to $\frac{1}{3}$ of the sum possessed by the other three, would make \$36; what sum had each? *Ans.* A, \$12; B, \$15; C, \$18; D, \$21.

14. The sum of three fractions is 2. The second fraction is double that of the first; and the third is double that of the second; what are the fractions? *Ans.* $\frac{2}{7}$, $\frac{4}{7}$, and $\frac{8}{7}$.

15. The first of three numbers with $\frac{1}{3}$ of the other two make 23; the second with $\frac{1}{2}$ of the other two make 30; the third with twice the sum of the other two make 72. What are the numbers? *Ans.* 12, 15, 18.

16. A's age is double that of B's, B's is triple that of C's, and the sum of all their ages is 140; what is the age of each?

Ans. A's $=84$; B's $=42$; C's $=14$.

17. A man wrought 10 days for his neighbor, his wife 4 days, and their son 3 days, and they all received 11 dollars 50 cents; at another time the man served 9 days, his wife 8 days,

and the son 6 days, at the same rates as before, and received 12 dollars; a third time the man served 7 days, his wife 6 days, and the son 4 days, at the same rates as before, and received 9 dollars. What were the daily wages of each?

Ans. Husband's wages, \$1.00; wife's, 0; son's, 50 cents.

NEGATIVE RESULTS.

157. In the solution of the last example, the wages of the wife are found to be 0, which means that she received no wages. The following examples will illustrate *negative* results:

1. A man worked for a person 10 days, having his wife with him 8 days and his son 6 days, and he received 10 dollars 30 cents as compensation for all three; at another time he wrought 12 days, his wife 10 days, and son 4 days, and he received 13 dollars and 20 cents; and at another time he wrought 15 days, his wife 10 days, and his son 12 days, at the same rates as before, and he received 13 dollars 85 cents. What were the daily wages of each?

Ans. Husband, 75 cents; wife, 50 cents; son, — 20 cents.

The sign *minus* signifies the opposite to the sign *plus.* Hence the son, instead of receiving *wages*, was at an *expense* of 20 cents a day, and the language of the problem is thus shown to be incorrect.

2. Two men, A and B, commenced trade at the same time; A had 3 times as much money as B, and continuing in trade, A gained 400 dollars, and B 150 dollars; A then had twice as much money as B. How much did each have at first?

Without any special consideration of the problem, it implies that both had money, and asks how much. But on solving the problem with x to represent A's money, and y B's, we find

$$x = -300$$

$$\text{and } y = -100 \text{ dollars.}$$

That is, they had no money, and the minus sign in this case indicates *debt;* and the solution not only reveals the numerical values, but the true conditions of the problem, and points out the necessary corrections of language to correspond to an arithmetical sense.

The problem should have been written thus:

A *was three times as much in debt as* B*;* A *gains* 400 *dollars, and* B 150; A *now has twice as much money as* B. *How much were each in debt?*

Ans. A's debt, \$300; B's, \$100.

These results are positive, and show that the enunciation corresponds to the real circumstances of the case.

3. What number is that whose fourth part exceeds its third part by 12? *Ans.* —144.

But there is no such abstract number as —144, and we cannot interpret this as *debt.* It points out error or *impossibility,* and by returning to the problem we perceive that a fourth part of any number whatever cannot exceed its third part; it must be, its third part exceeds its fourth part by 12, and the enunciation should be thus:

What number is that whose third part exceeds its fourth part by 12. *Ans.* 144.

Thus do equations rectify *subordinate* errors, and point out special conditions.

4. A man when he was married was 30 years old, and his wife 15. How many years must elapse before his age will be three times the age of his wife? *Ans.* —$7\frac{1}{2}$ years.

The question is incorrectly enunciated; $7\frac{1}{2}$ years *before* the marriage, *not* after, their ages bore the specified relation.

5.* What fraction is that which becomes $\frac{3}{5}$ when 1 is added to its numerator, and $\frac{5}{7}$ when 1 is added to its denominator.

Ans. In an arithmetical sense, there is no such fraction. The algebraic expression, $\frac{-10}{-15}$, will give the required results.

SECTION III.

INVOLUTION;

OR, THE FORMATION OF POWERS.

158. A **Power** is the product obtained by repeating a quantity several times as a factor.

159. Powers are indicated by exponents, from which they take their names.

Thus, let a represent any quantity:

Its *first* power is	$a = a^1$
Its *second* power is	$aa = a^2$
The *third* power is	$aaa = a^3$
The *fourth* power is	$aaaa = a^4$
The *fifth* power is	$aaaaa = a^5$

In general terms, a to the nth power is aa, &c., to n factors, and n may be any number whatever.

160. The **First Power** of any quantity is the quantity itself.

The **Square** of any quantity is its second power.

The **Cube** of any quantity is its third power.

161. A **Perfect Power** is a quantity that can be exactly produced by taking some other quantity a certain number of times as a factor; thus, $a^2 + 2ab + b^2$ is a perfect power, because it is equal to $(a + b) \times (a + b)$; $x^3 + 3x^2 + 3x + 1$ is a perfect power, because it is equal to $(x + 1)(x + 1)(x + 1)$.

NOTE.—It is thought best to omit questions at the bottom of the pages in the remaining part of this work, leaving the teacher to use such as may be deemed appropriate.

162. An **Imperfect Power** is a quantity that cannot be exactly produced by taking another quantity any number of times as a factor; as, $a^2 + b$, $x + 3y$, and $a^2 + ab + b^2$.

163. **Involution** is the process of raising any quantity to any given power. Involution, in algebra, is performed by successive multiplications, as in arithmetic.

The first power is the quantity itself.

The second power is the product of the quantity multiplied by itself.

The third power is the product of the second power by the quantity.

The fourth power is the third power multiplied by the quantity, etc.

POWERS OF MONOMIALS.

164. In the power of a monomial there are three things to be considered: 1st, the coefficient; 2d, the exponents; 3d, the sign.

1st. With respect to the coefficient:

Let it be required to raise $2a$ to the third power: we have

$$2a \times 2a \times 2a = 2 \times 2 \times 2a^3 = 2^3a^3 = 8a^3.$$

Hence, *The coefficient may be raised to the required power separately.*

2d. With respect to the exponents:

We observe that

The *second* power of a^3 is $a^3 \times a^3 = a^{3+3} = a^6$.

The *third* power of a^3 is $a^3 \times a^3 \times a^3 = a^{3+3+3} = a^9$.

The nth power of a^3 is $a^3 \times a^3 \times a^3 \times$ etc. $= a^{3+3+3+\text{etc.}}$ $= a^{3n}$.

Hence, *The exponent is repeated as many times as there are units in the index of the power.*

3d. With respect to the law of signs:

It is obvious that the repetition of any positive quantity as a factor must produce a positive result. But the successive powers of negative quantities must have varying signs.

Let it be required to raise $-a$ to successive powers. We have

Second power, $-a \times -a = +a^2$, positive.
Third power, $+a^2 \times -a = -a^3$, negative.
Fourth power, $-a^3 \times -a = +a^4$, positive.
Fifth power, $+a^4 \times -a = -a^5$, negative.

Hence,

1st. *All the powers of a positive quantity are positive.*

2d. *The even powers of a negative quantity are positive, and the odd powers negative.*

165. From these principles we deduce the following

RULE. I. *Raise the numeral coefficient to the required power.*

II. *Multiply the exponent of each letter by the index of the required power.*

III. *When the quantity is negative, give the odd powers the minus sign.*

EXAMPLES FOR PRACTICE.

1. Raise x^2 to the 3d power. *Ans.* x^6.
2. Raise y^5 to the 4th power. *Ans.* y^{20}.
3. Raise P^7 to the 5th power. *Ans.* P^{35}.
4. Raise x^3 to the 4th power. *Ans.* x^{12}.
5. Raise y^7 to the 3d power. *Ans.* y^{21}.
6. Raise x^n to the 6th power. *Ans.* x^{6n}.
7. Raise x^n to the mth power. *Ans.* x^{mn}.
8. Raise ax^2 to the 3d power. *Ans.* a^3x^6.
9. Raise ab^2x^4 to the 2d power. *Ans.* $a^2b^4x^8$.
10. Raise c^2y^4 to the 5th power. *Ans.* $c^{10}y^{20}$.
11. Required the 3d power of $3ax^2$. *Ans.* $27a^3x^6$.
12. Required the 3d power of $-2x$. *Ans.* $-8x^3$.
13. Required the 4th power of $-3x$. *Ans.* $81x^4$.
14. Required the 2d power of $8a^2b^3$. *Ans.* $64a^4b^6$.
15. Required the 3d power of $5x^2z$ *Ans.* $125x^6z^3$.

Expand the following indicated powers.

16. $(-2a^n)^5$. *Ans.* $-32a^{5n}$.
17. $(-a^2bc^3)^4$. *Ans.* $a^8b^4c^{12}$.
18. $(6a^5y^2x)^3$. *Ans.* $216a^{15}y^6x^3$.
19. $(2a^2b^3c^4)^4$. *Ans.* $16a^8b^{12}c^{16}$.
20. $(-3a^{2n}b^m)^3$. *Ans.* $-27a^{6n}b^{3m}$.
21. $(3m^mb^c)^5$. *Ans.* $243m^{5m}b^{5c}$.
22. $(-a^{-2})^4$. *Ans.* a^{-8}.
23. $(2b^{-m}c^{-n})^3$. *Ans.* $8b^{-3m}c^{-3n}$.
24. $(-c^{-2m}d^m)^5$. *Ans.* $-c^{-10m}d^{5m}$

POWERS OF FRACTIONS.

166. 1. What is the 3d power of $\frac{a}{c}$?

OPERATION.*

$$\left(\frac{a}{c}\right)^3 = \frac{a}{c} \times \frac{a}{c} \times \frac{a}{c} = \frac{a \times a \times a}{c \times c \times c} = \frac{a^3}{c^3}.$$

Hence, to raise fractions to powers, we have the following

RULE. *Raise both numerator and denominator to the required power.*

---

* Suppose we were required to raise $\frac{a}{b}$ to the fifth power, and did not know whether the denominator was to be raised or not, we could decide the point by means of an equation, as follows:

The fraction has *some value*, which we represent by a symbol, say P. Then $P = \frac{a}{b}$. Now if we can find the true 5th power of P, it will be the required 5th power of the fraction.

Clearing the equation of fractions, we have

$$bP = a$$

Taking the 5th power of both members gives

$$b^5P^5 = a^5.$$

By division, $$P^5 = \frac{a^5}{b^5}.$$

This equation shows that to raise any fraction to any power, the numerator and denominator must be raised to that power.

EXAMPLES FOR PRACTICE.

2. Required the 2d power of $\frac{2a^7b^6}{5c}$. *Ans.* $\frac{4a^{14}b^{12}}{25c^2}$.

3. Required the 6th power of $-\frac{2a}{3x}$. *Ans.* $\frac{64a^6}{729x^6}$.

4. Required the 6th power of $\frac{a^2b}{\frac{1}{3}x}$. *Ans.* $\frac{729a^{12}b^6}{x^6}$.

5. Required the 6th power of $\frac{2}{3}a^2b$. *Ans.* $\frac{64}{729}a^{12}b^6$.

6. Required the 2d power of $\frac{3}{a^2}$. *Ans.* $\frac{9}{a^4}$.

Expand the following indicated powers.

7. $\left(\frac{ac}{x^2y}\right)^3$. *Ans.* $\frac{a^3c^3}{x^6y^3}$.

8. $\left(-\frac{2y}{5x}\right)^4$. *Ans.* $\frac{16y^4}{625x^4}$.

9. $\left(\frac{ab}{3y}\right)^3$. *Ans.* $\frac{a^3b^3}{27y^3}$.

10. $\left(\frac{-a}{2c}\right)^5$. *Ans.* $\frac{-a^5}{32c^5}$.

11. $\left(\frac{c^2}{-y^2}\right)^3$. *Ans.* $\frac{c^6}{-y^6}$.

12. $\left(\frac{-3x^2y}{4ab^2}\right)^4$. *Ans.* $\frac{81x^8y^4}{256a^4b^8}$.

13. $\left(-\frac{a^{-m}b^2}{x^my^n}\right)^5$. *Ans.* $-\frac{a^{-5m}b^{10}}{x^{5m}y^{5n}}$.

14. What is the cube of $\frac{a^2-bc}{3x}$?

Ans. $\frac{a^6-3a^4bc+3a^2b^2c^2-b^3c^3}{27x^3}$.

15. What is the square of $\frac{2a+b+x}{-c-2d}$?

Ans $\frac{4a^2+4ab+4ax+b^2+2bx+x^2}{c^2+4cd+4d^2}$.

POWERS OF A BINOMIAL.

167. The **Leading Letter, Quantity, or Term,** is the one which is written first in the binomial; the other is called the *second*, or *following* letter, quantity, or term.

The process of expanding the higher powers of a binomial by actual multiplication is very tedious, and hence mathematicians long since sought to discover some shorter method. Such a method was first developed by Sir Isaac Newton, and is known as

NEWTON'S BINOMIAL THEOREM.

168. In order to more clearly investigate the properties of different powers of a binomial, we will first obtain a few powers of $a + b$ by continued multiplication; thus,

Let $a + b$ be raised to the 2d, 3d, 4th, &c., powers.

$$\begin{array}{ll}
 & a + b \\
 & \underline{a + b} \\
 & a^2 + ab \\
 & \underline{\quad\; ab + b^2} \\
\text{2d power,} & a^2 + 2ab + b^2 \\
 & \underline{a + b} \\
 & a^3 + 2a^2b + ab^2 \\
 & \underline{\quad\; a^2b + 2ab^2 + b^3} \\
\text{3d power,} & a^3 + 3a^2b + 3ab^2 + b^3 \\
 & \underline{a + b} \\
 & a^4 + 3a^3b + 3a^2b^2 + ab^3 \\
 & \underline{\quad\; a^3b + 3a^2b^2 + 3ab^3 + b^4} \\
\text{4th power,} & a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \\
 & \underline{a + b} \\
 & a^5 + 4a^4b + 6a^3b^2 + 4a^2b^3 + ab^4 \\
 & \underline{\quad\; a^4b + 4a^3b^2 + 6a^2b^3 + 4ab^4 + b^5} \\
\text{5th power,} & a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5
\end{array}$$

If $a - b$ be raised to the same powers, we have:

$$
\begin{array}{ll}
 & a-b \\
 & \underline{a-b} \\
 & a^2-ab \\
 & \underline{\quad -ab+b^2} \\
\text{2d power,} & a^2-2ab+b^2 \\
 & \underline{a-b} \\
 & a^3-2a^2b+\ ab^2 \\
 & \underline{\quad -\ a^2b+2ab^2-b^3} \\
\text{3d power,} & a^3-3a^2b+3ab^2-b^3 \\
 & \underline{a-b} \\
 & a^4-3a^3b+3a^2b^2-\ ab^3 \\
 & \underline{\quad -\ a^3b+3a^2b^2-3ab^3+b^4} \\
\text{4th power,} & a^4-4a^3b+6a^2b^2-4ab^3+b^4 \\
 & \underline{a-b} \\
 & a^5-4a^4b+6a^3b^2-4a^2b^3+\ ab^4 \\
 & \underline{\quad -\ a^4b+4a^3b^2-6a^2b^3+4ab^4-b^5} \\
\text{5th power,} & a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-b^5
\end{array}
$$

By inspecting these results we may arrive at general principles, according to which any power of a binomial may be expressed, without the labor of actual multiplication. In order to do this it is obvious that there are five things to be considered:

1*st.* The number of terms; 2*d,* The signs of the terms; 3*d,* The letters; 4*th,* The exponents; 5*th,* The coefficients

1*st.* The number of terms

169. We observe that in the *second* power there are *three* terms; in the *third* power there are *four* terms; in the *fourth* power, *five* terms; and in the *fifth* power, *six* terms. Hence,

The number of terms is always greater by one than the index of the power.

2*d*. The signs of the terms:

170. We see that all the terms in the powers of $a + b$ are positive; but in the powers of $a - b$, the signs *plus* and *minus* alternate, the first term being positive, the second negative, and so on. Hence,

I. *If both terms of the binomial have the plus sign, all the terms of any power will be positive.*

II. *But if the second term of the binomial have the minus sign, all the odd terms, counting from the left, will be positive, and all the even terms negative.*

3*d*. The letters:

171. By inspecting any power, we perceive that,

The second letter or quantity does not appear in the first term; the leading letter or quantity does not appear in the last term; and both letters or quantities appear in all the intermediate terms.

4*th*. The exponents:

172. We observe that in the fifth power of both binomials, the exponents of the letters in the several terms are related as follows, from the first to the last:

Of the leading letter,	5	4	3	2	1	
Of the second letter,		1	2	3	4	5
Sum,	5	5	5	5	5	5

Hence,

I. *The exponents of the leading letter or quantity in the successive terms form a series, commencing in the first term with the index of the power, and diminishing by* 1 *to the right.*

II. *The exponents of the second letter or quantity form a series commencing in the second term with* 1, *and increasing*

by 1 *to the last term, in which the exponent is equal to the index of the power.*

III. *The sum of the exponents in any term is equal to the index of the power.*

5th. The coefficients:

173. The law governing the coefficients, though less obvious, may be exhibited as follows, taking the 5th power:

1st term is $1a^5$, and $1 \times 5 = 5$, coefficient for 2d term;

2d term is $5a^4b$, and $\dfrac{5 \times 4}{2} = 10$, coefficient for 3d term;

3d term is $10a^3b^2$, and $\dfrac{10 \times 3}{3} = 10$, coefficient for 4th term;

4th term is $10a^2b^3$, and $\dfrac{10 \times 2}{4} = 5$, coefficient for 5th term;

5th term is $5ab^4$, and $\dfrac{5 \times 1}{5} = 1$, coefficient for 6th term;

6th term is $1b^5$.

Hence,

I. *The coefficient of the first term is* 1.

II. *The coefficient of the second term is the index of the required power.*

III. *The coefficient of any term multiplied by the exponent of the leading letter or quantity, and divided by the exponent of the second letter or quantity plus* 1, *will be the coefficient of the next succeeding term.*

NOTES. 1. It will be seen that the coefficients of the last half of the terms are the same as the coefficients of the first half inversely, and that the coefficients of any two terms at equal distances from the extremes are equal. Hence the labor of computing them may be avoided.

2. In obtaining the coefficient of any term, there are two operations, multiplication and division, and *cancellation* can always be applied.

3. The exponent of the second letter or quantity, plus 1, is always equal to the number of the term, counting from the left.

NOTE.—We have now established by induction, and observations upon particular cases, the principles of Newton's Theorem. Its rigid demonstration is somewhat difficult, but its application is simple and practical

EXAMPLES FOR PRACTICE.

1. Expand $(x+y)^3$. *Ans.* $x^3+3x^2y+3xy^2+y^3$.

2. Expand $(y+z)^7$.
Ans. $y^7+7y^6z+21y^5z^2+35y^4z^3+35y^3z^4+21y^2z^5+7yz^6+z^7$.

3. Expand $(a+b)^8$.
Ans. $a^8+8a^7b+28a^6b^2+56a^5b^3+70a^4b^4+56a^3b^5+28a^2b^6+8ab^7+b^8$.

4. Expand $(a-b)^4$.
Ans. $a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

5. Expand $(x+y)^6$.
Ans. $x^6+6x^5y+15x^4y^2+20x^3y^3+15x^2y^4+6xy^5+y^6$.

6. Expand $(x-y)^6$.
Ans. $x^6-6x^5y+15x^4y^2-20x^3y^3+15x^2y^4-6xy^5+y^6$.

7. Expand $(a+b)^{10}$.
Ans. $a^{10}+10a^9b+45a^8b^2+120a^7b^3+210a^6b^4+252a^5b^5+210a^4b^6+120a^3b^7+45a^2b^8+10ab^9+b^{10}$.

8. Expand $(ac+x)^4$.
Ans. $a^4c^4+4a^3c^3x+6a^2c^2x^2+4acx^3+x^4$.

9. Expand $(c+ax)^5$.
Ans. $c^5+5c^4ax+10c^3a^2x^2+10c^2a^3x^3+5ca^4x^4+a^5x^5$.

10. Expand $(ab+cxy)^3$.
Ans. $\overline{ab}^3+3\overline{ab}^2cxy+3ab\overline{cxy}^2+\overline{cxy}^3$.

11. Expand $(a+1)^5$.

NOTE.—Notice that all powers of 1 are 1, which is not written when a factor; and that the *divisor* in obtaining the coefficients will be the number of the term employed counted from the left.

Ans. $a^5+5a^4+10a^3+10a^2+5a+1$.

12. Expand $(1-a)^5$.
Ans. $1-5a+10a^2-10a^3+5a^4-a^5$.

13. Expand $(z-1)^6$.
Ans. $z^6-6z^5+15z^4-20z^3+15z^2-6z+1$.

14. Required the third power of $3x+2y$.

We cannot well expand this by the binomial theorem, because the terms are not simple *literal quantities*. But we can assume $3x = a$ and $2y = b$. Then

$3x + 2y = a + b$, and $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.

Now restoring the values of a and b, we have,

$$\begin{aligned} a^3 &= && 27x^3 \\ 3a^2b &= 3 \times 9x^2 \times 2y &&= 54x^2y \\ 3ab^2 &= 3 \times 3x \times 4y^2 &&= 36xy^2 \\ b^3 &= && 8y^3 \end{aligned}$$

Hence, $(3x + 2y)^3 = 27x^3 + 54x^2y + 36xy^2 + 8y^3$.

15. Required the 4th power of $2a^2 - 3$.

Let $x = 2a^2$, $y = 3$. Then expand $(x - y)^4$, and restore the values of x and y, and the result will be,

$$16a^8 - 96a^6 + 216a^4 - 216a^2 + 81.$$

16. Required the cube of $(a + b + c + d)$.

As we can operate in this summary manner *only* on *binomial* quantities, we represent $a + b$ by x, or assume $x = a + b$, and $y = c + d$.

Then $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$.

Restoring the values of x and y, we have

$$(a + b)^3 + 3(a + b)^2(c + d) + 3(a + b)(c + d)^2 + (c + d)^3.$$

Now we can expand the binomial quantities contained in parentheses.

The method of substitution which we have been obliged to adopt in the last three examples, has been long in use among mathematicians for expanding binomials with coefficients.

174. FRENCH'S THEOREM.

Any binomial having coefficients, may be involved by actual multiplication.

Required the 5th power of $2a + 3x$.

$$
\begin{array}{llllll}
 & 2a + 3x & & & & \\
 & 2a + 3x & & & & \\
\text{2d power,} & 4a^2 + 12ax & + 9x^2 & & & \\
 & 2a + 3x & & & & \\
 & 8a^3 + 24a^2x & + 18ax^2 & & & \\
 & \quad\;\; 12a^2x & + 36ax^2 & + 27x^3 & & \\
\text{3d power,} & 8a^3 + 36a^2x & + 54ax^2 & + 27x^3 & & \\
 & 2a + 3x & & & & \\
 & 16a^4 + 72a^3x & + 108a^2x^2 & + 54ax^3 & & \\
 & \quad\;\; 24a^3x & + 108a^2x^2 & + 162ax^3 & + 81x^4 & \\
\text{4th power,} & 16a^4 + 96a^3x & + 216a^2x^2 & + 216ax^3 & + 81x^4 & \\
 & 2a + 3x & & & & \\
 & 32a^5 + 192a^4x & + 432a^3x^2 & + 432a^2x^3 & + 162ax^4 & \\
 & \quad\;\; 48a^4x & + 288a^3x^2 & + 648a^2x^3 & + 648ax^4 & + 243x^5 \\
\text{5th power,} & 32a^5 + 240a^4x & + 720a^3x^2 & + 1080a^2x^3 & + 810ax^4 & + 243x^5.
\end{array}
$$

175. By a close analysis of the result, we may arrive at general principles which will enable us to expand any binomial having coefficients with the same facility that Newton's Theorem enables us to expand to any power $a + x$. If we examine the result in the same order as we did in (**168–172**), we shall find that there is no difference in the expanded form of $(a + x)^5$ and $(2a + 3x)^5$, except in the coefficients. That is,

In any power of a binomial, the number of terms, the signs, the letters, and the exponents of the literal part in the several terms are independent of the coefficients of the binomial root

We will therefore confine our analysis to

The Coefficients.

176. The rigid demonstration of the law which governs the formation of the coefficients being too difficult for this place, is reserved for the University Algebra. But the law itself may ...ibited as follows :

coefficient, $32 = 2^5$, *i. e.* the coefficient of the leading quantity in the root raised to the power of the given index.

coef., $240 = \frac{32 \times 5 \times 3}{2}$, *i. e.* the product of the coefficient of the first term, the exponent of the leading quantity in the first term, and the coefficient of the following quantity in the root, divided by the coefficient of the leading quantity in the root.

coef. $720 = \frac{240 \times 4 \times 3}{2 \times 2}$, *i. e.* the product of the coefficient of the second term, the exponent of a in that term, and the coefficient of x in the root, divided by the product of the coefficient of a in the root and the number of the term, counting from the left.

coef., $1080 = \frac{720 \times 3 \times 3}{2 \times 3}$, *i. e.* the product of the coefficient of the third term, the exponent of a in that term, and the coefficient of x in the root, divided by the product of the coefficient of a in the root and the number of the term. (Or, which is the same thing, the exponent of x in the third term + 1.)

5th coef., $810 = \frac{1080 \times 2 \times 3}{2 \times 4}$, *i. e.* the product of the coefficient of the fourth term, the exponent of a in that term, and the coefficient of x in the root, divided by the product of the coefficient of a in the root and the number of the term.

6th coef., $243 = \frac{810 \times 1 \times 3}{2 \times 5}$, *i. e.* the product of the coefficient of the fifth term, the exponent of a in that term, and the coefficient of x in the root, divided by the product of the coefficient of a in the root and the number of the term.

177. From this example and analysis we may deduce the

LAW OF THE COEFFICIENTS.

I. *The coefficient of the first term in any power is always equal to the corresponding power of the coefficient of the leading term in the root.*

II. *The coefficient of the second term is obtained by multiplying the first coefficient by the exponent of the leading quantity, and this product by the coefficient of the following quantity in the root, and dividing by the coefficient of the leading quantity in the root.*

UNIVERSALLY; — *The coefficient of any term may be obtained by multiplying the coefficient of the preceding term by the exponent of the leading quantity in that term, or by the number of the term from the last, and by the coefficient of the following quantity in the root, and dividing this result by the product of the coefficient of the leading quantity in the root, multiplied by the number of the term from the first.*

NOTE.—The coefficient of the last term in any power is always equal to the corresponding power of the coefficient of the following term in the root.

178. There is another class of binomials that come under a modification of this Theorem, viz.: those having exponents. To illustrate the application, let us write out the fourth power of $2a^3 + 3x^2$.

$$\begin{array}{llllll}
 & 2a^3 + 3x^2 & & & & \\
 & 2a^3 + 3x^2 & & & & \\
\hline
\text{2d power,} & 4a^6 & + 12a^3x^2 & + 9x^4 & & \\
 & 2a^3 & + 3x^2 & & & \\
\hline
 & 8a^9 & + 24a^6x^2 & + 18a^3x^4 & & \\
 & & 12a^6x^2 & + 36a^3x^4 & + 27x^6 & \\
\hline
\text{3d power,} & 8a^9 & + 36a^6x^2 & + 54a^3x^4 & + 27x^6 & \\
 & 2a^3 & + 3x^2 & & & \\
\hline
 & 16a^{12} & + 72a^9x^2 & + 108a^6x^4 & + 54a^3x^6 & \\
 & & 24a^9x^2 & + 108a^6x^4 & + 162a^3x^6 & + 81x^8 \\
\hline
\text{4th power,} & 16a^{12} & + 96a^9x^2 & + 216a^6x^4 & + 216a^3x^6 & + 81x^8
\end{array}$$

On examining the result, we shall find that the exponent of the leading letter is $12(=4 \times 3)$ in the first term, and that it diminishes regularly by 3 in each succeeding term. Also that the exponent of the following letter is 2 in the second term, and increases regularly by 2, in each succeeding term, to the last, where it is $8(=2 \times 4)$. Hence the exponents are governed by the law of Newton's Theorem, as shown in (**172**), modified by the values of the exponents.

The coefficients are the same as the coefficients of $(2a + 3x)^4$, (**174**), and may be obtained in the same manner, if we keep constantly in mind the fact that the first exponent, 12, is the exponent 3 of the leading quantity in the root raised to the fourth power, and that the real exponent which we are to use as a factor of our dividend is the exponent of the leading quantity in any term divided by the exponent of the leading quantity

in the root. But, as this is liable to be forgotten, we can use the exponent of the leading quantity in any term, whatever it may be, as a factor of the dividend, if we write the exponent of the leading quantity in the root, as a factor of the divisor. Observing this direction, and the indicated operations for obtaining the several coefficients in $(2a^3 + 3x^2)^4$, will be as follows:

1st coefficient, $2^4 = 16$

2d coefficient, $\dfrac{16 \times 12 \times 3}{3 \times 2} = 96$

3d coefficient, $\dfrac{96 \times 9 \times 3}{2 \times 3 \times 2} = 216$

4th coefficient, $\dfrac{216 \times 6 \times 3}{3 \times 3 \times 2} = 216$

5th coefficient, $\dfrac{216 \times 3 \times 3}{4 \times 3 \times 2} = 81$

179. Examining the indicated operations for obtaining the coefficients of the expanded form of $(2a + 3x)^5$ (**177**), we observe the following facts:

1*st.* Each dividend after the first term is composed of three factors, the first of which is the coefficient of the preceding term, the second, the exponent of the leading quantity in the preceding term, and the third, the coefficient of the following quantity in the root.

2*d.* Each divisor is composed of two factors, the first of which is the number of the preceding term counted from the left, and the second the coefficient of the leading quantity in the root.

3*d.* The second factor of the dividend decreases regularly by 1, and the first factor of the divisor increases regularly by 1, in each succeeding coefficient.

4*th.* The third factor of the dividend, and the second factor of the divisor, are the same in each coefficient, *i. e.*, they are constant.

We may, therefore,

Let $a =$ 1st coefficient of any binomial.
$b =$ 2d coefficient of any binomial.
$n =$ the exponent of any binomial.

Then $(ax \pm by)^n$ any power of any binomial.

Assume the first coefficient to be C_1, the second C_2, the third, C_3, &c., and we shall have the following

General Formula for Coefficients.

$$C_1 = a^n$$

$$C_2 = \frac{C_1 nb}{a}$$

$$C_3 = \frac{C_2(n-1)b}{2a}$$

$$C_4 = \frac{C_3(n-2)b}{3a}$$

$$C_5 = \frac{C_4(n-3)b}{4a}$$

&c. &c.

NOTE. We have now carried the investigation of this Theorem as far as the plan and limits of this work admit. It is general in its application, and may be used in the involution of any binomial whatever. Its full development, including its application to negative indices and binomial roots, will be found in future editions of the University Algebra.

EXAMPLES FOR PRACTICE.

1. Required the 4th power of $2a + 3x$.
Ans. $16a^4 + 96a^3x + 216a^2x^2 + 216ax^3 + 81x^4$.

2. Expand $(2a - 5b)^3$.
Ans. $8a^3 - 60a^2b + 150ab^2 - 125b^3$.

3. What is the cube of $7x + 2ay$?
Ans. $343x^3 + 294x^2ay + 84xa^2y^2 + 8a^3y^3$.

4. What is the fifth power of $5a - 2c$?
Ans. $3125a^5 - 6250a^4c + 5000a^3c^2 - 2000a^2c^3 + 400ac^4 - 32c^5$.

5. Expand $(x^2 + 3y^2)^5$.
Ans. $x^{10} + 15x^8y^2 + 90x^6y^4 + 270x^4y^6 + 405x^2y^8 + 243y^{10}$.

6. Expand $(2a^2 + ax)^3$.

Ans. $8a^6 + 12a^5x + 6a^4x^2 + 2a^3x^3$.

7. Expand $(x - 1)^6$.

Ans. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$.

8. Expand $(3x - 5)^3$.

Ans. $27x^3 - 135x^2 + 225x - 125$.

9. Expand $(4a^3b - 2c^2)^4$.

Ans. $256a^{12}b^4 - 512a^9b^3c^2 + 384a^6b^2c^4 - 128a^3bc^6 + 16x^8$.

10. Expand $\left(\frac{a}{2} + \frac{3x}{4}\right)^5$.

NOTE.—The quantity $\left(\frac{a}{2} + \frac{3x}{4}\right)^5 = \left(\frac{1}{2}a + \frac{3}{4}x\right)^5$.

Ans. $\frac{1}{32}a^5 + \frac{15}{64}a^4x + \frac{90}{128}a^3x^2 + \frac{270}{256}a^2x^3 + \frac{405}{512}ax^4 + \frac{243}{1024}x^5$.

11. Expand $(3 - 2r)$ to the 6th power.

Ans. $729 - 2916r + 4860r^2 - 4320r^3 + 2160r^4 - 576r^5 + 64r^6$.

12. Expand $\left(x + \frac{1}{2x}\right)$ to the 7th power.

Ans. $x^7 + \frac{7}{2}x^5 + \frac{21}{4}x^3 + \frac{35}{8}x + \frac{35}{16x} + \frac{21}{32x^3} + \frac{7}{64x^5} + \frac{1}{128x^7}$.

13. Expand $\left(1 + \frac{5}{2}x\right)^5$.

Ans. $1 + \frac{25}{2}x + \frac{125}{2}x^2 + \frac{625}{4}x^3 + \frac{3125}{16}x^4 + \frac{3125}{32}x^5$.

14. Expand $\left(\frac{3}{2} - \frac{5}{3}x\right)^4$.

Ans. $\frac{81}{16} - \frac{45}{2}x + \frac{75}{2}x^2 - \frac{250}{9}x^3 + \frac{625}{81}x^4$.

15. Expand $(x^2 - 3y^2)^5$.

Ans. $x^{10} - 15x^8y^2 + 90x^6y^4 - 270x^4y^6 + 405x^2y^8 - 243y^{10}$.

EVOLUTION;

OR, THE EXTRACTION OF ROOTS.

180. A **Root** is a factor repeated to form a power; or, it is one of the equal factors of a quantity.

181. Evolution is the process of extracting the root of a quantity. It is the converse of involution, and is indicated by the radical sign, $\sqrt{}$.

182. The **Index** of the root is the figure placed above the radical sign to denote what root of the quantity under the radical is to be taken; thus, in $\sqrt[5]{a}$, 5 is the index of the root, and denotes that the fifth root of a is to be taken.

183. A **Surd** is the indicated root of an imperfect power; the root thus indicated cannot be exactly obtained or expressed; thus, $\sqrt{2}$ is a surd, because the number 2, not being a perfect square, can have no exact square root. A surd is sometimes called an *irrational* quantity.

ROOTS OF MONOMIALS.

184. To discover the process of extracting roots, we must *observe* how powers are formed, and then *trace the operations back.* Thus, to square a, we double its exponent, which makes a^2. (**57.**) The square of a^2 is a^4, the cube of a^2 is a^6, &c. The 4th power of x, is x^4; the nth power of x^4 is x^{4n}; &c., &c.

Now, since *multiplying exponents raises simple literal quantities to powers, dividing exponents must extract roots.* Thus, the square root of a^4 is $a^{4\div2} = a^2$; the cube root of a^9 is $a^{9\div3} = a^3$.

The square root of a must have its exponent (1 understood), divided by 2, which will give $a^{\frac{1}{2}}$; the cube root of a in the like manner is $a^{\frac{1}{3}}$, and the exponents, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, &c., indicate the

second, third, fourth, and fifth roots of any quantity whose exponent is 1. The 6th root of x^5 is $x^{\frac{5}{6}}$. In like manner, $\frac{3}{4}$ expresses the 4th root of the 3d power of a quantity. Hence the following principles:

I. *Roots are properly expressed by fractional exponents.*

II. *The numerator shows the power of the quantity, whose root is to be extracted.*

III. *The denominator shows what root of that power is to be extracted. It is the index of the root.*

We have seen (**164**) that any power of a positive quantity is positive, and that the even powers of a negative quantity are positive, and the odd powers negative. From this it results, that

I. *The odd roots of a positive quantity are always positive, and the even roots are either positive or negative.*

II. *The odd roots of a negative quantity are negative, and the even roots are impossible or imaginary.*

NOTE.—An *Imaginary Quantity* is the indicated *even root* of a negative quantity, as $\sqrt{-a}$, or $\sqrt[4]{-2ab}$.

1. What is the square root of $64a^4x^2$?

OPERATION.

$$(64a^4x^2)^{\frac{1}{2}} = \pm 8a^2x$$

VERIFICATION.

$$(+8a^2x) \times (+8a^2x) = 64a^4x^2$$
$$(-8a^2x) \times (-8a^2x) = 64a^4x^2$$

ANALYSIS. Since the power of a monomial is formed by involving each factor, (**165**), conversely, the root may be obtained by extracting the root of each factor separately. The square root of 64 is 8; of a^4 is a^2; of x^2 is x; and the entire root is $8a^2x$, to which we give the double sign, $\pm$, (read *plus* or *minus*), because either $+8a^2x$, or $-8a^2x$, squared, will produce $64a^4x^2$, as is seen in the verification.

185. From these principles and illustrations we have the following

RULE. I. *Extract the required root of the numeral coefficient.*

II. *Divide the exponent of each letter by the index of the root.*

III. *Prefix the double sign, $\pm$, to all even roots, and the minus sign to the odd roots of a negative quantity.*

NOTE 1. Under this rule for monomials we shall introduce no numeral coefficient the required root of which will consist of more than one place; hence the root may be found by *trial.*

EXAMPLES FOR PRACTICE.

1. What is the *second* root of $9a^2x^4y^6$? *Ans.* $\pm 3ax^2y^3$.
2. What is the *third* root of $8a^6y^3$? *Ans.* $2a^2y$.
3. What is the *fourth* root of $81a^4x^{12}$? *Ans.* $\pm 3ax^3$.
4. What is the *fifth* root of $32a^5x^{10}y^{15}$? *Ans.* $2ax^2y^3$.
5. What is the *fourth* root of $81a^4b^8c^{12}$? *Ans.* $\pm 3ab^2c^3$.
6. What is the *third* root of $-27a^{12}x^3$? *Ans.* $-3a^4x$.

Find the following indicated roots:

7. $(-27a^6b^3)^{\frac{1}{3}}$. *Ans.* $-3a^2b$.
8. $(25x^8y^6)^{\frac{1}{2}}$. *Ans.* $\pm 5x^4y^3$.
9. $\sqrt[4]{16x^{4m}}$. *Ans.* $\pm 2x^m$.
10. $\sqrt[5]{a^2y^3}$. *Ans.* $a^{\frac{2}{5}}y^{\frac{3}{5}}$.
11. $\sqrt[3]{125a^6m^2}$. *Ans.* $5a^2m^{\frac{2}{3}}$, or $5a^2\sqrt[3]{m^2}$.
12 $\sqrt[n]{a^{2n}}$. *Ans.* a^2.
13. $(x^my^2)^{\frac{1}{m}}$. *Ans.* $xy^{\frac{2}{m}}$, or $x\sqrt[m]{y^2}$.
14. $(a^mb^n)^{\frac{1}{n}}$. *Ans.* $a^{\frac{m}{n}}b$, or $b\sqrt[n]{a^m}$.
15. Find the cube root of $4a^6$.

NOTE 2 If the coefficient is an imperfect power, it may be treated as a literal factor, and its root *indicated.*

Ans. $4^{\frac{1}{3}}a^2$, or $a^2\sqrt[3]{4}$.

16. Find the 5th root of $7a^2b^{10}$. *Ans.* $7^{\frac{1}{5}}a^{\frac{2}{5}}b^2$.

17. Find the 9th root of $-15x^3y^{\frac{1}{2}}$. *Ans.* $-15^{\frac{1}{9}}x^{\frac{1}{3}}y^{\frac{1}{18}}$.

18. Extract the square root of $\frac{a^2}{4b^4}$.

NOTE 3. Since the power of a fraction is formed by involving the numerator and denominator separately, the root of a fraction will be obtained by extracting the root of the numerator and denominator separately.

Ans. $\pm\frac{a}{2b^2}$.

19. Extract the cube root of $-\frac{27x^3}{8a^6y^3}$. *Ans.* $-\frac{3x}{2a^2y}$.

20. Extract the fourth root of $\frac{m}{n^4}$. *Ans.* $\pm\frac{m^{\frac{1}{4}}}{n}$.

21. Extract the square root of $\frac{64a^8}{81x^2}$. *Ans.* $\pm\frac{8a^4}{9x}$.

SQUARE ROOT OF POLYNOMIALS.

186. In order to discover the process of extracting the square root of a polynomial, we must observe how the squares of polynomials are formed. If we square $a + b$, we shall have

$$(a + b)^2 = a^2 + 2ab + b^2.$$

This result, expressed in words, is as follows:

The square of the first term, plus twice the product of the two terms, plus the square of the second term.

The last two terms of the power may be factored as follows:

$$2ab + b^2 = (2a + b)b,$$

which is expressed thus:

Twice the first term plus the second, multiplied by the second.

1. Extract the square root of $a^2 + 2ab + b^2$.

OPERATION.

$$\begin{array}{r|l} & a^2 + 2ab + b^2 \;(\,a + b \\ & a^2 \\ \hline 2a + b\,) & 2ab + b^2 \\ & 2ab + b^2 \end{array}$$

ANALYSIS. Reversing the process of involution, we extract the square root of a^2, and obtain a, the first term of the root. The next term of the power is $2ab = 2a \times b$, or *twice the first term of the root multiplied by the second*, we therefore divide this term by $2a$, twice the first term of the root, to obtain b, the second term of the root. Placing b in the divisor also, at the right of $2a$, we have $2a + b$, or *twice the first term plus the second*, which, multiplied by b, gives $2ab + b^2$, the last two terms of the power.

Again, let us form the square of any polynomial, as $a + b + c$, in the following manner:

$$\begin{aligned} \text{Assume } s &= a + b, \text{ the first part.} \\ c &= \quad\quad \text{the second part.} \\ \text{Then } (s + c)^2 &= s^2 + 2sc + c^2. \text{ Hence,} \end{aligned}$$

The square of any polynomial, considered in two parts, is equal to the square of the first part, plus twice the product of the two parts, plus the square of the second part.

Thus the root of any quantity can be brought into a binomial, and the rule for a binomial root will answer for a root containing any number of terms, by considering the root *already found*, however great, as *one term*, or *one part.*

2. Find the square root of $a^2 + 2ab + b^2 + 2ac + 2bc + c^2$.

OPERATION.

$$\begin{array}{l|l} & a^2 + 2ab + b^2 + 2ac + 2bc + c^2 (a + b + c) \\ & a^2 \\ \hline 2a + b & \quad 2ab + b^2 \\ & \quad 2ab + b^2 \\ \hline 2a + 2b + c & \quad\quad\quad 2ac + 2bc + c^2 \\ & \quad\quad\quad 2ac + 2bc + c^2 \end{array}$$

Analysis.—Proceeding as before, we obtain two terms of the root, $a + b$, and a remainder of $2ac + 2bc + c^2$. We now consider $a + b$ as the *first part* of the required root, and write $2a + 2b$, or *twice the part already found*, for a divisor. Dividing, we obtain c, the next term of the root, which, as before, we place in both the root and divisor. Multiplying this complete divisor by c, and subtracting the product from the dividend, we have no remainder, and the work is complete.

187. From these illustrations we deduce the following

Rule. I. *Arrange the terms according to the powers of some letter, beginning with the highest, and write the square root of the first term in the root.*

II. *Subtract the square of the root thus found from the first term, and bring down the next two terms for a dividend.*

III. *Divide the first term of the dividend by twice the root already found, and write the result both in the root and in the divisor.*

IV. *Multiply the divisor, thus completed, by the term of the root last found, and subtract the product from the dividend, and proceed with the remainder, if any, as before.*

Note.—According to the principles established in (**184**), every square root obtained will still be a root, if all the signs of its terms be changed.

EXAMPLES FOR PRACTICE.

1. What is the square root of $a^4 + 4a^2b - 4a^2 + 4b^2 - 8b + 4$? *Ans.* $a^2 + 2b - 2$.

2. What is the square root of $1 - 4b + 4b^2 + 2y - 4by + y^2$? *Ans.* $1 - 2b + y$.

3. What is the square root of $4x^4 - 4x^3 + 13x^2 - 6x + 9$? *Ans.* $2x^2 - x + 3$.

4. What is the square root of $4x^4 - 16x^3 + 24x^2 - 16x + 4$? *Ans.* $2x^2 - 4x + 2$.

5. What is the square root of $16x^4 + 24x^3 + 89x^2 + 60x + 100$? *Ans.* $4x^2 + 3x + 10$.

6. What is the square root of $4x^4 - 16x^3 + 8x^2 + 16x + 4$?
Ans. $2x^2 - 4x - 2$.

7. What is the square root of $x^2 + 2xy + y^2 + 6xz + 6yz + 9z^2$?
Ans. $x + y + 3z$

8. What is the square root of $a^2 - ab + \frac{1}{4}b^2$?
Ans. $a - \frac{1}{2}b$.

9. What is the square root of $\frac{a^2}{b^2} - 2 + \frac{b^2}{a^2}$?
Ans. $\frac{a}{b} - \frac{b}{a}$, or $\frac{b}{a} - \frac{a}{b}$.

10. What is the square root of $x^{\frac{2}{3}} - 2x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$?
Ans. $x^{\frac{1}{3}} - y^{\frac{1}{3}}$, or $y^{\frac{1}{3}} - x^{\frac{1}{3}}$.

11. What is the square root of $1 - 4z + 10z^2 - 20z^3 + 25z^4 - 24z^5 + 16z^6$?
Ans. $1 - 2z + 3z^2 - 4z^3$.

12. What is the square root of $a^6 - 6a^5c + 15a^4c^2 - 20a^3c^3 + 15a^2c^4 - 6ac^5 + c^6$?
Ans. $a^3 - 3a^2c + 3ac^2 - c^3$.

13. What is the square root of $z^2 - 2z + 1 + 2zh - 2h + h^2$?
Ans. $z + h - 1$.

SQUARE ROOT OF NUMBERS.

188. In extracting the square root of numbers, the first thing to be considered is the relative number of places in a given number and its square root. This relation is exhibited in the following illustrations:

Roots.	Squares.	Roots.	Squares.
1	1	1	1
9	81	10	1,00
99	98,01	100	1,00,00
999	99,80,01	1000	1,00,00,00

From these examples we perceive that a number consisting of *one* place may have *one* or *two* places in the square; and that in all cases the addition of *one* place to the root adds *two* places to the square. Hence,

If a number be pointed off into periods of two figures each, commencing at the right, the number of full periods, and the left hand full or partial period will be equal to the number of places in the square root; the highest period answering to the highest figure of the root.

189. The square of any numeral quantity may be formed after the manner of algebraic squares.

For example, let $a = 40$, and $b = 7$; then $a + b = 47$. And since the square of $a + b$ will represent the square of 47, we have

$$\begin{array}{r}
a^2 = 1600 \\
2ab = 560 \\
b^2 = 49 \\
\hline
a^2 + 2ab + b^2 = 2209 = (47)^2.
\end{array}$$

Hence, the binomial square may be used as a formula for extracting the square roots of numbers.

1. Extract the square root of the number 2209.

OPERATION.

$$\begin{array}{rl|l}
 & 22{,}09 & 40 + 7 = 47 \\
a^2 = & 1600 & \\
\hline
2a = 80 & 609 & \\
2a + b = 87 & 609 & \\
\end{array}$$

ANALYSIS. Here are two periods indicating two places in the root, corresponding to tens and units. The greatest square in 22 is 16, its root is 4, or 4 tens $= 40$. Hence, $a = 40$. Then $2a = 80$, which we use as a divisor for 609, and obtain 7 for a quotient. The 7 is taken as the value of b, and $2a + b$, the complete divisor, is 87, which, multiplied by 7, gives the last two terms of the binomial square, $2ab + b^2 = 609$, and the entire root, $40 + 7 = 47$, is found.

Arithmetically, a may be taken as 4 instead of 40, and we may write 16 in hundreds' place, instead of 1600, the ciphers being superfluous. Then $2a$ will be 8 instead of 80, and in dividing, we say 8 is contained in 60 (not in 609) 7 times.

If the given number consists of more than two periods, we obtain the two superior figures of the root from the first two

periods, as before, and bring down another period to the remainder. We then consider the root already found as one quantity, and treat it as one figure.

2. What is the square root of 399424 ?

OPERATION.

	39,94 24(632
	36
123	394
	369
1262	25 24
	25 24

ANALYSIS. Disregarding the local value of the figures, we have $a = 6$, $2a = 12$, and 12 in 39, 3 times, which gives $b = 3$. We next suppose $a = 63$, and $2a = 126$; and 126 in 252, 2 times, or the second value of $b = 2$. In the same manner, we would repeat the formula of a binomial square as many times as we have periods. It is evident that we may obtain the divisor 126 from the last complete divisor 123 simply by doubling its last figure 3; and thus the divisors may be derived each from the next preceding, successively.

From these examples and illustrations we deduce the following

RULE. I. *Point the given number off into periods of two figures each, counting from the units' place to the left and right.*

II. *Find the greatest perfect square in the left-hand period, and write its root for the first figure in the required root; subtract the square of this figure from the first period, and to the remainder bring down the next period for a dividend.*

III. *Double the root already found, and write the result on the left for a divisor; find how many times this divisor is contained in the dividend, exclusive of the right-hand figure, and place the result in the root and at the right of the divisor.*

IV. *Multiply the divisor thus completed by the last figure of the root; subtract the product from the dividend; and to the remainder bring down the next period for a new dividend.*

V. *Double the right-hand figure of the last complete divisor for a new divisor, and continue the operation as before.*

EXAMPLES FOR PRACTICE.

3. What is the square root of 8836? *Ans.* 94.

4. What is the square root of 106929? *Ans.* 327.

5. What is the square root of 4782969? *Ans.* 2187.

6. What is the square root of 43046721? *Ans.* 6561.

7. What is the square root of 387420489? *Ans.* 19683.

8. What is the square root of 1209996225? *Ans.* 34785.

9. What is the square root of 6596038656? *Ans.* 81216.

10. What is the square root of 342694144? *Ans.* 18512.

11. What is the square root of 2573733569796? *Ans.* 1604286.

12. What is the square root of 10.4976? *Ans.* 3.24.

13. What is the square root of 3271.4207? *Ans.* 57.19 +.

14. What is the square root of 4795.25731? *Ans.* 69.247 +.

15. What is the square root of .0036? *Ans.* .06.

16. What is the square root of .00032754? *Ans.* .01809 +.

17. What is the square root of .00103041? *Ans.* .0321.

NOTE.—If both terms of a fraction are perfect squares, or if the fraction can be reduced to terms which are squares, the root may be obtained by the rule for algebraic fractions. Otherwise, the fraction may be reduced to a decimal.

18. What is the square root of $\frac{25}{81}$? *Ans.* $\frac{5}{9}$.

19. What is the square root of $\frac{49}{361}$? *Ans.* $\frac{7}{19}$.

20. What is the square root of $\frac{7225}{17689}$? *Ans.* $\frac{85}{133}$.

21. What is the square root of $\frac{72}{128}$?

Observe $\frac{72}{128} = \frac{36}{64}$. Hence, the square root is $\frac{6}{8}$.

22. What is the square root of $\frac{3304}{7434}$? *Ans.* $\frac{2}{3}$.

23. What is the square root of $\frac{2704}{4225}$? *Ans.* $\frac{4}{5}$.

24 What is the square root of $\frac{3}{4}$? *Ans.* .866 +.

25. What is the square root of $\frac{7}{9}$? *Ans.* .8819 +.

CUBE ROOT OF POLYNOMIALS.*

190. We may derive the method of extracting the cube root of an algebraic quantity in a manner similar to that pursued in square root, by analyzing and retracing the combination of terms in the binomial cube. Forming the cube of $a + b$, we have

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3;$$

from which we see that

I. *The first term of the power is the cube of the first term of the root;* and

II. *The second term of the power is three times the square of the first term of the root multiplied by the second.*

It is evident, therefore, that to find the first term of the root, we must extract the cube root of the first term of the power; and to find the second term of the root, we must divide the second term of the power, $3a^2b$, by three times the square of the first term of the root, $3a^2$; thus,

$$3a^2b \div 3a^2 = b.$$

The last three terms of the power may be factored as follows:

$$(3a^2 + 3ab + b^2)b.$$

To reproduce these terms from the divisor already found and the root, we must complete our partial divisor, $3a^2$, by the addi-

* We are indebted to J. C. Porter, A.M., of the Clinton Liberal Institute, for the valuable method of Cube Root presented here and in the Practical Arithmetic. It is an extension of and improvement upon Horner's Method, and secures the result with less labor than any other method heretofore presented.

tion of $3ab + b^2$, and multiply the divisor thus completed by b. Putting the correction, $3ab + b^2$, under the form of $(3a + b)b$, we shall have,

$$\begin{aligned} 3a^2 &= \text{Trial divisor.} \\ 3a \quad + b &= \text{First factor of correction.} \\ 3ab + b^2 &= \text{Correction of trial divisor.} \\ 3a^2 + 3ab + b^2 &= \text{Complete divisor.} \end{aligned}$$

1. Find the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$.

OPERATION.

$$\begin{array}{ll|ll} & & & a^3 + 3a^2b + 3ab^2 + b^3 \,\underline{|\, a + b} \\ & & & a^3 \\ \hline & & 3a^2 & 3a^2b + 3ab^2 + b^3 \\ 3a + b & 3ab + b^2 & 3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 \\ \hline \end{array}$$

ANALYSIS. Taking the cube root of a^3, we obtain a, the first term of the root. Subtracting the cube of a from the given polynomial, we have $3a^2b + 3ab^2 + b^3$ for a remainder or dividend. We next write $3a^2$ at the left of the dividend for a trial divisor. Dividing the first term of the dividend, we obtain b, the second term of the root. We next multiply the former term of the root by 3, and annex the latter term, b, and obtain $3a + b$, the first factor of the correction to the trial divisor. Multiplying this by b, we have $3ab + b^2$, the correction. Adding this to the trial divisor, we have $3a^2 + 3ab + b^2$, the complete divisor. Multiplying the complete divisor by the last term of the root, and subtracting the result from the dividend, we have no remainder, and the work is complete.

Again, let us form the cube of any polynomial, as $a + b + c$, in the following manner:

$$\begin{aligned} \text{Assume } s &= a + b, \text{ the first part;} \\ c &= \qquad\quad \text{the second part.} \\ \text{Then } (s + c)^3 &= s^3 + 3s^2c + 3sc^2 + c^3. \end{aligned}$$

The first two terms of the root, regarded as one part, sustain the same relation to the third, as the first sustains to the second; and so on.

The binomial cube, therefore, furnishes the method of extracting any cube root whatever, by treating the root *already found*, at each step, as a *simple term*.

2. What is the cube root of $x^6 - 40x^3 + 6x^5 + 96x - 64$?

OPERATION.

			$x^6+6x^5-40x^3+96x-64$	x^2+2x-4
			x^6	
		$3x^4$	$6x^5-40x^3+96x-64$	
$3x^2+2x$	$6x^3+4x^2$	$3x^4+6x^3+4x^2$	$6x^5+12x^4+8x^3$	
		$3x^4+12x^3+12x^2$	$-12x^4-48x^3+96x-64$	
$3x^2+6x-4$	$-12x^2-24x+16$	$3x^4+12x^3-24x+16$	$-12x^4-48x^3+96x-64$	

ANALYSIS. Since it was shown, in involution, that the exponents of any letter in a power form a regular series, we arrange the terms according to the powers of x. The cube root of x^6 is x^2, the first term of the root; subtracting the cube of x^2 from the polynomial, and arranging the remainder according to the powers of x, we have $6x^5 - 40x^3 + 96x - 64$ for a dividend. We next write 3 times the square of x^2, or $3x^4$, for a trial divisor; and dividing $6x^5$, the first term of the dividend, we obtain $2x$ for the second term of the root. Having found the second term, we must complete our divisor as in the first example. Therefore, to 3 times the first term we annex the second, and obtain $3x^2 + 2x$, the first factor; and multiplying this by the second term, we have $6x^3 + 4x^2$ for the correction to the trial divisor. Adding, we have $3x^4 + 6x^3 + 4x^2$, the complete divisor. Multiplying this by the second term, $2x$, and subtracting the product from the dividend, we have for a new dividend, $-12x^4 - 48x^3 + 96x - 64$.

Now, since the two terms of the root already found, considered as one part, sustain the same relation to the third term, as the first term sustains to the second, the trial divisor to obtain the third term will be 3 times the square of the first two terms, or $3(x^2 + 2x)^2 = 3x^4 + 12x^3 + 12x^2$. This quantity is found in the operation by adding together $4x^2$, the square of the last term of the root; $6x^3 + 4x^2$, the correction; and $3x^4 + 6x^3 + 4x^2$, the first complete divisor. Dividing $-12x^4$, the first term of the dividend, by $3x^4$, the first term of the divisor, we obtain -4, for the third term of the root.

To find a correction of the trial divisor, the first factor will be the last term, -4, annexed to three times the former terms of the root, or $3x^2 + 6x - 4$. This quantity is found in the operation by taking the first factor of the last correction, with its last term multiplied by 3, and annexing the -4. Multiplying this by -4, we obtain $-12x^2 - 24x + 16$, for the correction. Adding this to the trial divisor, we have $3x^4 + 12x^3 - 24x + 16$, for the complete divisor. Multiplying this by -4, and subtracting the result from the dividend, we have no remainder, and the work is complete.

191. From these illustrations we deduce the following

RULE. I. *Arrange the polynomial according to the powers of some letter, and write the cube root of the first term in the root.*

II. *Subtract the cube of the root thus found from the polynomial, and arrange the remainder for a dividend.*

III. *At the left of the dividend write three times the square of the root already found for a trial divisor; divide the first term of the dividend by this divisor, and write the quotient for the next term of the root.*

IV. *To three times the first term of the root annex the last term, and write the result at the left, and, one line below, the trial divisor; multiply this binomial factor by the last term of the root, for a correction to the trial divisor; add the correction, and the result will be the complete divisor.*

V. *Multiply the complete divisor by the last term of the root, subtract the product from the dividend, and arrange the remainder for a new dividend.*

VI. *Add together the square of the last term of the root, the last correction, and the last complete divisor, for a new trial divisor, and by division obtain another term of the root.*

VII. *Take the first factor of the last correction with its last term multiplied by 3, and annex to it the last term of the root, for the first factor of the correction to the new trial divisor, with which proceed as in the former steps, till the work is completed.*

NOTE.—The *first term* of the remainder, when properly arranged, will be that term which contains the highest power of the *leading letter* of the root, or of the arranged polynomial.

EXAMPLES FOR PRACTICE.

1. What is the cube root of $8 + 12a + 6a^2 + a^3$?

Ans. $2 + a$.

2. What is the cube root of $27a^3 + 108a^2 + 144a + 64$?

Ans. $3a + 4$.

3. What is the cube root of $a^3 - 6a^2x + 12ax^2 - 8x^3$?

Ans. $a - 2x$

4. What is the cube root of $x^6 - 3x^5 + 5x^3 - 3x - 1$?

Ans. $x^2 - x - 1$.

5. What is the cube root of $a^3 - 6a^2b + 12ab^2 - 8b^3$?

Ans. $a - 2b$.

6. What is the cube root of $x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}$?

Ans. $x + \frac{1}{x}$.

7. What is the cube root of $x^9 + 3x^8 + 6x^7 + 10x^6 + 12x^5 + 12x^4 + 10x^3 + 6x^2 + 3x + 1$?

Ans. $x^3 + x^2 + x + 1$.

8. What is the cube root of $a^9 - 3a^8 + 8a^6 - 6a^5 - 6a^4 + 8a^3 - 3a + 1$?

Ans. $a^3 - a^2 - a + 1$.

CUBE ROOT OF NUMBERS.

192. To apply the binomial cube as a formula for the extraction of the cube root of numbers, we must first ascertain the relative number of places in a cube and its root. This relation will be seen in the following examples.

Roots.	Cubes.	Roots.	Cubes.
1	1	1	1
9	729	10	1,000
99	970,299	100	1,000,000
999	997,002,999	1000	1,000,000,000

From these illustrations, we perceive that a number consisting of *one* place, may have from *one to three* places in its cube; and that in all cases the addition of *one* place to the root adds *three* places to the cube. Hence,

If a number be pointed off into three-figure periods, commencing at the right, the number of full periods, and the

left-hand full or partial period, will indicate the number of places in its cube root; the highest period answering to the highest figure of the root.

193. To form the cube of a number, let $a = 50$, and $b = 4$. Then $a + b = 54$; and cubing, we have,

$$\begin{aligned} a^3 &= 125000 \\ 3a^2b &= 30000 \\ 3ab^2 &= 2400 \\ b^3 &= 64 \end{aligned}$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = 157464 = (54)^3$$

Hence, in the cube of a number,

The figures of the root, with their local values, have the same combinations as the terms of an algebraic quantity.

1. What is the cube root of 157464?

OPERATION.

			157,464	54
			125	
		7500	32464	
154	616	8116	32464	

ANALYSIS. Pointing off the number, the two periods show that there will be two figures, tens and units, in the root. Since the highest figure of the root corresponds to the highest period of the power, we find the greatest perfect cube in the first or left hand period, which is 125, and place 5, its root, for the figure of the required root. Subtracting the cube number 125 from the first period, and bringing down the next period, we have 32464 for a remainder or dividend. Since the figures in a cube root, with their local values, have the same combinations in the cube as the terms of an algebraic quantity, we write at the left of the dividend three times the square of the root already found, or 75, with two ciphers annexed, for a trial divisor. Dividing, we obtain 4 for the next figure of the root. To complete the divisor, we multiply the first figure of the root by 3, and annex the last, and obtain 154 for the first factor of the correction. Multiplying this number by 4, we have 616, the correction to the trial divisor. Adding, we have 8166, the complete divisor. And multiplying this by 4, and subtracting the product from the dividend, there is no remainder, and the work *is complete.*

2. What is the cube root of 12812904?

OPERATION.

		12,812,904	234
		8	
		1200	4812
63	189	1389	4167
		158700	645904
694	2776	161476	645904

ANALYSIS. The greatest cube in the first period is 8, and its root is 2, which we write for the first figure of the required root. Subtracting 8, and bringing down the next period, we have 4812 for a dividend. Annexing two ciphers to 3 times the square of 2, we have 1200 for a trial divisor. Dividing, we obtain 3, the next figure of the root. To complete the divisor, we have, by the same method as before, 63 for the first factor of the correction, and 189 for the correction. Adding the correction, we obtain 1389 for the complete divisor. Multiplying this by 3, subtracting the product from the dividend, and bringing down the next period, we have 645904 for a new dividend. As in the algebraic method, we add 9, the square of the last root figure; 189, the last correction; and 1389, the last complete divisor; and annex two ciphers, for a new trial divisor. Dividing, we obtain 4, the next figure of the root. We then take the first factor of the last correction, with its last figure multiplied by 3, and annex the last root figure, 4, and obtain 694 for the first factor of the new correction. Multiplying this by 4, we have 2776, the correction. Then completing the divisor, multiplying by the last root figure, and subtracting the product from the dividend, we have no remainder, and the work is complete.

From these examples we derive the following

RULE. I. *Point off the given number into periods of three figures each, counting from units' place to the left and right.*

II. *Find the greatest cube in the left-hand period, and place its root for the first figure of the required root. Subtract the cube from the first period, and to the remainder bring down the next period for a dividend.*

III. *At the left of the dividend, write three times the square of the root already found, and annex two ciphers for*

a trial divisor; divide the dividend, and write the quotient for the next term of the root.

IV. *To three times the first figure of the root annex the last, and place the result at the left, and one line below the trial divisor; multiply the factor by the last root figure, for a correction to the trial divisor; add the correction, and the result will be the complete divisor.*

V. *Multiply the complete divisor by the last figure of the root, subtract the product from the dividend, and to the remainder bring down another period for a new dividend.*

VI. *Add together the square of the last figure of the root, the last correction, and the last complete divisor, and annex two ciphers for a new trial divisor; and by division obtain another figure of the root.*

VII. *Take the first factor of the last correction, with its unit figure multiplied by 3, and annex the last figure of the root, for the first factor of the correction to the new trial divisor, with which proceed as in the former steps till the work is complete.*

NOTES. 1. If at any time the product is greater than the dividend, diminish the corresponding root figure and correct the erroneous work.

2. If a cipher occur in the root, annex two more ciphers to the trial divisor, bring down another period in the dividend, and proceed as before.

EXAMPLES FOR PRACTICE.

3. What is the cube root of 148877? *Ans.* 53
4. What is the cube root of 571787? *Ans.* 83.
5. What is the cube root of 1367631? *Ans.* 111.
6. What is the cube root of 2048383? *Ans.* 127.
7. What is the cube root of 16581375? *Ans.* 255.
8. What is the cube root of 44361864? *Ans.* 354.
9. What is the cube root of 100544625? *Ans.* 465.
10. What is the cube root of 12358435328? *Ans.* 2312.

11. What is the cube root of 999700029999? *Ans.* 9999.

12. What is the cube root of 2456? *Ans.* 13.491+.

13. What is the cube root of .004019679? *Ans.* .159.

14. What is the cube root of 2287.148? *Ans.* 13.175+.

CONTRACTED METHOD.

194. The methods of direct extraction of the cube root of surd numbers are all too tedious to be much used, and several eminent mathematicians have given more brief and practical methods of approximation.

One of the most useful methods may be investigated as follows:

Suppose a and $a + c$ two cube roots, c being *very small* in relation to a; a^3 and $a^3 + 3a^2c + 3ac^2 + c^3$ are the cubes of the supposed roots.

Now, if we double the first cube (a^3), and add it to the second, we shall have

$$3a^3 + 3a^2c + 3ac^2 + c^3.$$

If we double the second cube and add it to the first, we shall have

$$3a^3 + 6a^2c + 6ac^2 + 2c^3.$$

As c is a very small fraction compared to a, the terms containing c^2 and c^3 are very small in relation to the others; and the relation of these two sums will not be materially changed by rejecting those terms containing c^2 and c^3, and the sums will then be

$$3a^3 + 3a^2c$$

And $$3a^3 + 6a^2c.$$

The ratio of these terms is the same as the ratio of $a + c$ to $a + 2c$.

Or the ratio is $$1 + \frac{c}{a + c}.$$

But the ratio of the roots a to $a + c$, is $1 + \frac{c}{a}$.

Observing again, that c is supposed to be very small in relation to a, the fractional parts of the ratios $\frac{c}{a+c}$ and $\frac{c}{a}$ are both small, and very near in value to each other. Hence, we have found an operation on two cubes which are near each other in magnitude, and that will give results very *near* in proportion to their roots; and by knowing the root of one of the cubes, by this ratio we can find the other. And as this relation will still exist if one of the roots is a surd, the proportion will furnish a method of approximating to values of surds.

For example, let it be required to find the cube root of 28, true to 4 or 5 places of decimals. Since 27 is a cube near in value to 28, the root of which we know to be 3,

$$\text{Assume} \quad a^3 = 27, \quad \text{or} \quad a = 3.$$
$$(a+c)^3 = 28, \quad \text{or } a + c = \sqrt[3]{28}.$$

$$\begin{array}{lcc}
\text{Then} & 27 & 28 \\
 & 2 & 2 \\
\hline
 & 54 & 56 \\
\text{Add} & 28 & 27 \\
\hline
\text{Sums} & 82 & 83
\end{array}$$

$82 : 83 :: 3 : a + c$ very nearly.

Or, $(a + c) = \frac{249}{82} = 3.03658\dot{} +$, which is the cube root of 28, true to 5 places of decimals.

By the laws of proportion, which we hope more fully to investigate in a subsequent part of this work, the above proportion, $82 : 83 :: a : a + c$,
may take this form, $82 : 1 :: a : c$, c being a correction to the known root, a.

Hence $c = \frac{3}{82} = .03658 +$;
And $a + c = 3.03658 +$, as before.

195. From this investigation, we deduce the following rule for finding approximate cube roots:

Rule. *Take the nearest rational cube to the given number, or assume a root and cube it. Double this cube, ana*

add the number to it; also double the number and add the assumed cube to it. Then, by proportion, the first sum is to the second, as the known root is to the required root.

Or, *The first sum is to the difference of the two sums, as the known root is to a correction to the known root.*

EXAMPLES FOR PRACTICE.

1. What is the approximate cube root of 122?
   Ans. 4.95967 +.

2. What is the cube root of 10? *Ans.* 2.65441 +.

NOTE.—Assume 2.1 for the root, then 9.261 is its cube.

3. What is the approximate cube root of 720?
   Ans. 8.9628 +.

4. What is the approximate cube root of 345?
   Ans. 7.01357 +.

5. What is the approximate cube root of 520?
   Ans. 8.04145 +.

6. What is the approximate cube root of 65?
   Ans. 4.0207 +.

7. What is the approximate cube root of 16?

The cube root of 8 is 2, and of 27 is 3; therefore the cube root of 16 is between 2 and 3. Suppose it 2.5. The cube of this root is 15.625, which shows that the cube root of 16 is a little more than 2.5, and by the rule

15.625	16
2	2
31.250	32
16	15.625
47.25	47.625
	47.25

47.25 : .375 :: 2.5 : .01984

Assumed root,	2.50000
Correction,	.01984
Approximate root,	2.51984

We give the last as an example to be followed in most cases where the root is about midway between two integral numbers.

This method may be used with advantage to extract the root of perfect cubes, when very large, as will be seen in the examples which follow.

8. The number 22.069.810.125 is a cube; required its root.

ANALYSIS. Dividing this cube into periods, we find that the root must contain 4 figures, the superior period is 22, the cube root of 22 is near 3, and of course the whole root near 3000; but it is less than 3000. Suppose it 2800, and cube this number. The cube is 21952000000, which, being less than the given number, shows that our assumed root is not large enough.

To apply the rule, it will be sufficient to take six superior figures of the given and assumed cubes. Then by the rule,

219520	220698
2	2
439040	441396
220698	219520
659738	660916
	659738

659738 : 1178 :: 2800
2800
942400
2356
659738)3298400(5
3298690

Assumed root,	2800
Correction,	5
True root,	2805

The result of the last proportion is not exactly 5, as will be seen by inspecting the work; the slight imperfection arises from the rule being approximate, not perfect.

NOTE. — When we have cubes, we can always decide the unit figure by inspection, and, in the last example, the unit figure in the cube being 5, the unit figure in the root must be 5, as no other figure when cubed will give 5 in the place of units.

9. The number 41135081408 is a perfect cube; required its root. *Ans.* 3452.

10. The number 125525735343 is a perfect cube; required its root. *Ans.* 5007.

REDUCTION OF RADICALS.

196. A **Radical Quantity** is a root merely indicated by the radical sign or by a fractional exponent; as $2\sqrt{a}$, $5\sqrt[3]{a^2-2b}$, $c(3a^2b)^{\frac{1}{4}}$.

The quantity or factor placed before a radical is its coefficient. Thus, 2, 5, and c, in the above examples, are the coefficients of the radicals

197. The *degree* of a radical quantity is denoted by the radical index, or by the denominator of the fractional exponent. Thus,

$\sqrt{ab}$, $x^{\frac{1}{2}}$, $(a+b)^{\frac{1}{2}}$ are radicals of the 2d degree;

$\sqrt[3]{m}$, $\sqrt[3]{a^2}$, $(2x)^{\frac{1}{3}}$ are radicals of the 3d degree;

$\sqrt[n]{x^2}$, $y^{\frac{1}{n}}$, $(a-c)^{\frac{1}{n}}$ are radicals of the nth degree.

198. **Similar Radicals** are those having the same quantity under a radical sign of the same index. Thus, $3\sqrt[3]{a^2-b}$, $-\sqrt[3]{a^2-b}$, and $5(a^2-b)^{\frac{1}{3}}$, are similar radicals.

199. **Reduction of Radical Quantities** is the process of changing their forms without altering their values.

CASE I.

200. To reduce a radical to its simplest form.

A radical is in its *simplest form* when it contains no perfect powers corresponding to the degree of the radical. It has

been shown (**185**) that the root of a quantity is obtained by dividing the exponent of each factor by the index of the required root; by which process the root of each factor is taken separately. Hence,

The root of a quantity is equal to the product of the roots of its component factors.

1. Reduce $\sqrt{75a^2x}$ to its simplest form.

OPERATION.

$$\begin{aligned}\sqrt{75a^2x} &= \sqrt{25a^2 \times 3x}\\ &= \sqrt{25a^2} \times \sqrt{3x}\\ &= 5a\sqrt{3x}\end{aligned}$$

ANALYSIS. Since the radical is of the second degree, we separate the factors of the quantity under the radical sign into two groups, one of which, $25a^2$, contains all the *perfect squares*, and the other, $3x$, all the *surds* in the quantity. And since the root of a quantity is equal to the product of the roots of its component factors, we extract the square root of the rational part, $25a^2$, and obtain $5a$, and multiplying this result by the indicated root of the other part, we obtain $5a\sqrt{3x}$, the simplest form of the radical.

2. Reduce $5\sqrt[3]{a^4 - a^3b}$ to its simplest form.

OPERATION.

$$\begin{aligned}5\sqrt[3]{a^4 - a^3b} &= 5\sqrt[3]{a^3(a-b)}\\ &= 5\sqrt[3]{a^3} \times \sqrt[3]{a-b}\\ &= 5a\sqrt[3]{a-b}\end{aligned}$$

ANALYSIS. Since the radical is of the third degree, we separate the quantity under the sign into two factors, one of which, a^3, is a perfect cube. Taking the cube root of this factor, and multiplying this root, a, the coefficient, 5, and the surd, $\sqrt[3]{a-b}$, together, we have $5a\sqrt[3]{a-b}$, the simplest form of the radical.

From these illustrations we deduce the following

RULE. I. *Separate the factors of the quantity under the radical sign into two groups, one of which shall contain all the perfect powers corresponding in degree with the radical.*

II. *Extract the root of the rational part, and multiply the root, coefficient, and surd or radical part together.*

EXAMPLES FOR PRACTICE.

Reduce the following radicals to their simplest form,

3. $\sqrt{a^2bc}$. *Ans.* $a\sqrt{bc}$.

4. $2\sqrt{x^3y^2}$. *Ans.* $2xy\sqrt{x}$.

5. $3\sqrt{50x^5}$. *Ans.* $15x^2\sqrt{2x}$.

6. $a\sqrt[3]{16a^3b}$. *Ans.* $2a^2\sqrt[3]{2b}$.

7. $5\sqrt[4]{81m^5}$. *Ans.* $15m\sqrt[4]{m}$.

8. $\sqrt{a^2 - a^2c}$. *Ans.* $a\sqrt{1 - c}$.

9. $xy\sqrt{x^3y^2 - x^2y^3}$. *Ans.* $x^2y^2\sqrt{x - y}$.

10. $4\sqrt{72ab^2c^3}$. *Ans.* $24bc\sqrt{2ac}$.

11. $2a\sqrt{147a^5x^2y}$. *Ans.* $14a^3x\sqrt{3ay}$.

12. $5\sqrt[3]{125x}$. *Ans.* $25\sqrt[3]{x}$.

13. $2c\sqrt[5]{32ac^{11}}$. *Ans.* $4c^3\sqrt[5]{ac}$.

14. $(a + b)\sqrt{a^3 - 2a^2b + ab^2}$. *Ans.* $(a^2 - b^2)\sqrt{a}$.

15. $(a - b)\sqrt{a^2b + 2ab^2 + b^3}$. *Ans.* $(a^2 - b^2)\sqrt{b}$.

16. $d\sqrt{x^3y - 2x^2y^2 + xy^3}$. *Ans.* $d(x - y)\sqrt{xy}$.

17. $(180x^3y)^{\frac{1}{2}}$. *Ans.* $6x(5xy)^{\frac{1}{2}}$.

18. $(24x^5y^3z)^{\frac{1}{3}}$. *Ans.* $2xy(3x^2z)^{\frac{1}{3}}$.

19. $(54am^9)^{\frac{1}{3}}$. *Ans.* $3m^3(2a)^{\frac{1}{3}}$.

20. $(a^7z^3 - a^6bz^3)^{\frac{1}{3}}$. *Ans.* $a^2z(a - b)^{\frac{1}{3}}$

CASE II.

201. To reduce a rational quantity to a radical, or to introduce a coefficient of a radical under the radical sign.

1. Reduce $5ax^2$ to the form of the cube root.

OPERATION.

$$5ax^2 = (5ax^2)^{\frac{3}{3}} = \sqrt[3]{125a^3x^6}$$

ANALYSIS. We cube each factor of the given quantity separately, and indicate the cube root of the result

2. Reduce $2c\sqrt[4]{x}$ to a radical without a coefficient.

OPERATION.

$$2c = (2c)^{\frac{4}{4}} = (16c^4)^{\frac{1}{4}}$$
$$(16c^4)^{\frac{1}{4}} \times x^{\frac{1}{4}} = (16c^4x)^{\frac{1}{4}}$$
Hence $$2c\sqrt[4]{x} = \sqrt[4]{16c^4x}$$

ANALYSIS. We raise the coefficient, $2c$, to the fourth power, and we have $(16c^4)^{\frac{1}{4}}$. Multiplying this result by $x^{\frac{1}{4}}$, we have $\sqrt[4]{16c^4x}$. Hence, the

RULE. I. To reduce a rational quantity to a radical:— *Involve it to the same power as the required index, and write the result under the corresponding radical sign.*

II. To introduce a coefficient of a radical quantity under the radical:—*Involve it to the same power as the radical, multiply the radical by the result, and write the product under the radical sign.*

EXAMPLES FOR PRACTICE.

3 Reduce ax^2z^8 to the form of the square root. *Ans.* $\sqrt{a^2x^4z^{16}}$.

4. Reduce $9a^5y$ to the form of the cube root. *Ans.* $\sqrt[3]{729a^{15}y^3}$, or $(729a^{15}y^3)^{\frac{1}{3}}$.

5. Reduce $a + cx$ to the form of the fourth root. *Ans.* $(a^4 + 4a^3cx + 6a^2c^2x^2 + 4ac^3x^3 + c^4x^4)^{\frac{1}{4}}$.

6. Introduce the coefficient of $a^3\sqrt{c}$ under the radical sign. *Ans.* $\sqrt{a^6c}$.

7. Introduce the coefficient of $3a\sqrt[3]{2a^4x}$ under the radical sign. *Ans.* $\sqrt[3]{54a^7x}$.

Reduce the following quantities to equivalent radicals without coefficients:

8. $(2a - c)\sqrt[3]{4}$. *Ans.* $(32a^3 - 48a^2c + 24ac^2 - 4c^3)^{\frac{1}{3}}$.

9. $4c^3\sqrt[5]{ac}$. *Ans.* $\sqrt[5]{1024ac^{16}}$.

10. $ax^2(a + bxy)^{\frac{1}{4}}$. *Ans.* $\sqrt[4]{a^5x^8 + a^4bx^9y}$.

11. $(ab+x)\sqrt{a^2b^2-2abx+x^2}$.

Ans. $\sqrt{a^4b^4-2a^2b^2x^2+x^4}$.

12. $(a^2-b^2)\sqrt{a}$. *Ans.* $(a^5-2a^3b^2+ab^4)^{\frac{1}{2}}$.

CASE III.

202. To reduce radicals of different degrees to a common radical index.

1. Reduce $a^{\frac{1}{2}}$ and $b^{\frac{1}{3}}$ to a common radical index.

OPERATION.

$$a^{\frac{1}{2}}=a^{\frac{3}{6}}$$

$$b^{\frac{1}{3}}=b^{\frac{2}{6}}$$

$$a^{\frac{1}{2}}=(a^3)^{\frac{1}{6}},\ b^{\frac{1}{3}}=(b^2)^{\frac{1}{6}}$$

Or,

$$a^{\frac{1}{2}}=\sqrt[6]{a^3},\ b^{\frac{1}{3}}=\sqrt[6]{b^2}$$

ANALYSIS. We have seen (201) that the value of any quantity is not changed by involving it to any power and placing the result under the corresponding radical sign or index. That is, the nth root of the nth power of any quantity is the quantity itself. Now, as the index of a in the given example is $\frac{1}{2}$, and of b $\frac{1}{3}$, we may raise the two quantities to any powers that will make the denominators of their indices the same. This we do by reducing the indices $\frac{1}{2}$ and $\frac{1}{3}$ to equivalent indices having a common denominator, as shown in the operation. Hence,

RULE. I. *Reduce the indices to a common denominator.*

II. *Write the numerator of each equivalent index as an exponent of its respective quantity, and place the result under the common radical sign or index.*

EXAMPLES FOR PRACTICE.

2. Reduce $a^{\frac{2}{3}}$, $d^{\frac{1}{2}}$, and $c^{\frac{3}{n}}$ to a common radical index.

OPERATION.

$$\frac{2}{3},\ \frac{1}{2},\ \frac{3}{n}=\frac{4n}{6n},\ \frac{3n}{6n},\ \frac{18}{6n}.$$

$$a^{\frac{2}{3}}=a^{\frac{4n}{6n}}=\sqrt[6n]{a^{4n}}=(a^{4n})^{\frac{1}{6n}}$$

$$d^{\frac{1}{2}}=d^{\frac{3n}{6n}}=\sqrt[6n]{d^{3n}}=(d^{3n})^{\frac{1}{6n}}$$

$$c^{\frac{3}{n}}=c^{\frac{18}{6n}}=\sqrt[6n]{c^{18}}=(c^{18})^{\frac{1}{6n}}.$$

3. Reduce m, $(an)^{\frac{2}{3}}$, cxy^2, and 5 to a common radical index. *Ans.* $(m^3)^{\frac{1}{3}}$, $(a^2n^2)^{\frac{1}{3}}$, $(c^3x^3y^6)^{\frac{1}{3}}$, $(125)^{\frac{1}{3}}$.

4. Reduce a^4x, $(cm)^{\frac{2}{n}}$ and $\sqrt{d}$ to a common radical index.
Ans. $\sqrt[2n]{a^{8n}x^{2n}}$, $\sqrt[2n]{c^4m^4}$, $\sqrt[2n]{d^n}$.

ADDITION OF RADICALS.

203. 1. What is the sum of $3\sqrt{ab}$ and $5\sqrt{ab}$?

OPERATION.

$$\begin{array}{r} 3\sqrt{ab} \\ 5\sqrt{ab} \\ \hline 8\sqrt{ab} \end{array}$$

ANALYSIS. We make the common radical, $\sqrt{ab}$, the unit of addition; and adding the coefficients, we obtain $8\sqrt{ab}$, the required sum.

2. What is the sum of $\sqrt[3]{250a^5}$ and $\sqrt[3]{16a^5}$?

OPERATION.

$$\begin{array}{ll} \sqrt[3]{250a^5} = & 5a\sqrt[3]{2a^2} \\ \sqrt[3]{16a^5} = & 2a\sqrt[3]{2a^2} \\ \hline \text{Sum,} & 7a\sqrt[3]{2a^2} \end{array}$$

ANALYSIS. We reduce the radicals to their simplest form, and obtain two similar radicals. Adding their coefficients, we have $7a\sqrt[3]{2a^2}$.

From these examples we deduce the following

RULE. I. *Reduce each radical to its simplest form.*

II. *If the resulting radicals are similar, add their coefficients, and to the sum annex the common radical; if dissimilar, indicate the addition by the plus sign.*

EXAMPLES FOR PRACTICE.

3. Add $3\sqrt{3a^2x}$ and $a\sqrt{48x}$ together. *Ans.* $7a\sqrt{3x}$.

4 Add $\sqrt{80m}$ and $\sqrt{125m}$ together. *Ans.* $9\sqrt{5m}$.

5. Add $\sqrt{72}$, $\sqrt{128}$, and $\sqrt{8}$. *Ans.* $16\sqrt{2}$.

6. Add $x\sqrt{3ay^2}$, $3y\sqrt{3ax^2}$, and $2\sqrt{3ax^2y^2}$.
Ans. $6xy\sqrt{3a}$.

7. Find the sum of $\sqrt{2a^2xy}$ and $\sqrt{2b^2xy}$.
Ans. $(a+b)\sqrt{2xy}$.

8. Find the sum of $\sqrt{a^2x - a^2y}$ and $\sqrt{4a^2(x - y)}$.
Ans. $3a\sqrt{x-y}$.

9. Find the sum of $\sqrt{80a^2b^2}$ and $\sqrt{245a^8b^6}$.
Ans. $(4ab + 7a^4b^3)\sqrt{5}$.

10. Find the sum of $3\sqrt{3a^2x}$, $\sqrt{12a^2x}$, and $\sqrt{3b^2x}$.
Ans. $(5a + b)\sqrt{3x}$.

11. Find the sum of $\sqrt{2a^4}$, $2\sqrt{2a^2b^2}$, and $\sqrt{2b^4}$.
Ans. $(a + b)^2\sqrt{2}$.

12. Find the sum of $\sqrt[3]{8a^2}$ and $\sqrt[3]{125a^2}$.
Ans. $7\sqrt[3]{a^2}$.

13. Find the sum of $\sqrt[3]{270a^3m}$ and $\sqrt[3]{1250b^3m}$.
Ans. $(3a + 5b)\sqrt[3]{10m}$.

14. Find the sum of $\sqrt[3]{x^7y}$, $\sqrt[3]{8x^4y^4}$, and $\sqrt[3]{xy^7}$.
Ans. $(x+y)^2\sqrt[3]{xy}$.

15. Find the sum of $\sqrt{\frac{9a}{16}}$ and $\sqrt{\frac{a}{16}}$. *Ans.* $\sqrt{a}$.

16. Find the sum of $\sqrt{a^2b}$ and $\sqrt{ab^2}$
Ans. $a\sqrt{b} + b\sqrt{a}$.

17. Find the sum of $\sqrt{x^2m}$ and $\sqrt{x^2n}$.
Ans. $x(\sqrt{m} + \sqrt{n})$.

18. Find the sum of $2(4a^2b)^{\frac{1}{2}}$ and $(36a^2b)^{\frac{1}{2}}$.
Ans. $10ab^{\frac{1}{2}}$

19. Find the sum of $(a^3x^4 - a^3x^3y)^{\frac{1}{3}}$ and $a(x^4 - x^3y)^{\frac{1}{3}}$.
Ans. $2ax(x-y)^{\frac{1}{3}}$.

SUBTRACTION OF RADICALS.

204. 1. From $\sqrt{98a}$ take $\sqrt{50a}$.

OPERATION.

$\sqrt{98a} = 7\sqrt{2a}$
$\sqrt{50a} = 5\sqrt{2a}$
Difference, $2\sqrt{2a}$

ANALYSIS. Reducing the radicals to their simplest form, we obtain the two similar radicals, $7\sqrt{2a}$ and $5\sqrt{2a}$. Making the radical part the unit of subtraction, we take the difference of the coefficients, and obtain $2\sqrt{2a}$. Hence the following

RULE I. *Reduce each radical to its simplest form.*

II. *If the resulting radicals are similar, subtract the coefficient of the subtrahend from the coefficient of the minuend, and to the remainder annex the common radical; if dissimilar, indicate the subtraction by the minus sign.*

EXAMPLES FOR PRACTICE.

2. From $3\sqrt{5a^2c}$ take $a\sqrt{5c}$. *Ans.* $2a\sqrt{5c}$.
3. From $\sqrt{162x^4y}$ take $4\sqrt{8x^4y}$. *Ans.* $x^2\sqrt{2y}$.
4. From $\sqrt{20a^4b^3y}$ take $\sqrt{5a^4b^3y}$. *Ans.* $a^2b\sqrt{5by}$.
5. From $3\sqrt[3]{128a^3bc}$ take $4a\sqrt[3]{16bc}$. *Ans.* $4a\sqrt[3]{2bc}$.
6. From $\sqrt[3]{375a^4b}$ take $\sqrt[3]{24ab^4}$. *Ans.* $(5a-2b)\sqrt[3]{3ab}$.
7. From $3(16a^9b^2)^{\frac{1}{4}}$ take $2a(a^5b^2)^{\frac{1}{4}}$. *Ans.* $4a^2(ab^2)^{\frac{1}{4}}$.
8. From $\sqrt{3a^2c+6abc+3b^2c}$ take $\sqrt{12b^2c}$. *Ans.* $(a-b)\sqrt{3c}$.
9. From $\sqrt{2a^3c^2}$ take $\sqrt{2bc^2}$. *Ans.* $ac\sqrt{2a}-c\sqrt{2b}$.
10. From $\sqrt{a^3-a^2b}$ take $\sqrt{ab^2-b^3}$. *Ans.* $(a-b)\sqrt{a-b}$.
11. From $\sqrt{3}$ take $\sqrt{\frac{3}{4}}$. *Ans.* $\frac{1}{2}\sqrt{3}$.
12. From $6\sqrt[3]{32}$ take $6\sqrt[3]{\frac{4}{27}}$. *Ans.* $10\sqrt[3]{4}$.
13. From $2\sqrt[5]{32a^9}$ take $4\sqrt[5]{a^5b^4}$. *Ans.* $4a(\sqrt[5]{a^4}-\sqrt[5]{b^4})$.

MULTIPLICATION OF RADICALS.

CASE I.

205. To multiply radicals of the same degree.

Since the root of a quantity composed of several factors is obtained by extracting the root of each factor separately (**185**), we have

$$(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} \times b^{\frac{1}{n}};$$

Or, by radical sign, $\sqrt[n]{ab} = \sqrt[n]{a} \times \sqrt[n]{b}$;

Conversely, $\sqrt[n]{a} \times \sqrt[n]{b} = \sqrt[n]{ab}$.

Hence, if we consider a and b as representing any two quantities, and n the index of any root, we have

The product of the roots of any two quantities is equal to the root of their product.

1. Multiply $3a\sqrt{x}$ by $2\sqrt{y}$.

OPERATION.

$$3a\sqrt{x} \times 2\sqrt{y} = 6a\sqrt{xy}$$

ANALYSIS. Since the product will be the same, in whatever order the factors are taken, we multiply the coefficients $3a$ and 2, and obtain $6a$; and the radical parts $\sqrt{x}$ and $\sqrt{y}$, and obtain, by the principle enunciated above, $\sqrt{xy}$; and the entire product is $6a\sqrt{xy}$. Hence the following

RULE I. *Multiply the coefficients together for the coefficient of the product.*

II. *Multiply the quantities in the radical parts together, and place the product under the common radical sign.*

III. *Reduce the entire result to its simplest form.*

EXAMPLES FOR PRACTICE.

2. Multiply $2a\sqrt{3x}$ by $4\sqrt{y}$. *Ans.* $8a\sqrt{3xy}$.
3. Multiply $5\sqrt{ac}$ by $\sqrt{am}$. *Ans.* $5a\sqrt{mc}$.
4. Multiply $3\sqrt{5xy}$ by $4\sqrt{20x}$. *Ans.* $120x\sqrt{y}$.
5. Multiply $2\sqrt[3]{9x^2}$ by $\sqrt[3]{3xyz}$. *Ans.* $6x\sqrt[3]{yz}$.
6. Multiply $2\sqrt[3]{14}$ by $3\sqrt[3]{4}$. *Ans.* $12\sqrt[3]{7}$.
7. Multiply $3\sqrt{3}$ by $2\sqrt{3}$. *Ans.* 18.
8. Multiply $3\sqrt{2}$ by $4\sqrt{8}$. *Ans.* 48.
9. Multiply $\sqrt{6}$ by $\sqrt{150}$. *Ans.* 30.
10. Multiply $\sqrt{\frac{1}{2}}$ by $\sqrt{\frac{3}{8}}$. *Ans.* $\frac{1}{4}\sqrt{3}$.
11. Multiply $a + \sqrt{b}$ by $\sqrt{b}$. *Ans.* $a\sqrt{b} + b$.
12. Multiply $x + \sqrt{y}$ by $x - \sqrt{y}$. *Ans.* $x^2 - y$.
13. Multiply $\sqrt{m} + \sqrt{n}$ by $\sqrt{m} - \sqrt{n}$. *Ans.* $m - n$.
14. Multiply $\sqrt{a} + \sqrt{c}$ by $\sqrt{a} + \sqrt{c}$. *Ans.* $a + 2\sqrt{ac} + c$.
15. Multiply $a(b)^{\frac{1}{2}}$ by $c(d)^{\frac{1}{2}}$. *Ans.* $ac(bd)^{\frac{1}{2}}$.
16. Multiply $2c(a^2bd)^{\frac{1}{3}}$ by $(3ab)^{\frac{1}{3}}$. *Ans.* $2ac(3b^2d)^{\frac{1}{3}}$.
17. Multiply $(x^3y)^{\frac{1}{4}}$ by $(xy^3)^{\frac{1}{4}}$. *Ans.* xy.

CASE II.

206. To multiply radicals of different degrees.

1. What is the product of $a^{\frac{1}{2}}$ multiplied by $b^{\frac{1}{3}}$?

OPERATION.

$$P = a^{\frac{1}{2}}b^{\frac{1}{3}} \quad (1)$$
$$P^2 = ab^{\frac{2}{3}} \quad (2)$$
$$P^6 = a^3b^2 \quad (3)$$
$$P = (a^3b^2)^{\frac{1}{6}} \quad (4)$$

ANALYSIS. Letting P represent the product of the given quantities, we form equation (1). Squaring both members we have (2), in which the index of the factor a is 1. Cubing (2) we have (3), in which both factors are cleared of their radical indices; and extracting the sixth root, we have (4), in which the product is under a common index.

SECOND OPERATION.

$$a^{\frac{1}{2}} = a^{\frac{3}{6}}$$
$$b^{\frac{1}{3}} = b^{\frac{2}{6}}$$
$$a^{\frac{3}{6}}b^{\frac{2}{6}} = (a^3b^2)^{\frac{1}{6}}$$

ANALYSIS. We first reduce the given radicals to equivalent quantities having a common radical index, by (Case III., Reduction), and then multiplying $a^{\frac{3}{6}}$ by $b^{\frac{2}{6}}$ by Case I., we have the same result as before. Hence,

RULE. I. *Reduce the radical parts of the given quantities to a common radical index.*

II. *Multiply the rational and radical parts separately, as in Case I.*

EXAMPLES FOR PRACTICE.

2. Multiply $a^{\frac{1}{2}}$ by $a^{\frac{1}{3}}$. *Ans.* $a^{\frac{5}{6}}$.

3. Multiply $6^{\frac{1}{2}}$ by $(150)^{\frac{1}{2}}$. *Ans.* 30.

4. Give the product of $\sqrt{\frac{1}{2}}$, multiplied by $\sqrt[3]{\frac{3}{8}}$? *Ans.* $\frac{1}{2}\sqrt[6]{\frac{9}{8}}$.

5. Give the product of $2^{\frac{1}{2}}$ multiplied by $2^{\frac{1}{5}}$? *Ans.* $\sqrt[10]{128}$

6. Required the product of $(a + b)^{\frac{1}{3}}\ (a + b)^{\frac{1}{2}}$.
Ans. $(a + b)^{\frac{5}{6}}$.

7. Required the product of $4\sqrt{a} \times 3b\sqrt[n]{d + x}$.
Ans. $12b\sqrt[2n]{a^n(d + x)^2}$.

8. Multiply $\sqrt{\dfrac{a}{b}}$ by $\sqrt[4]{\dfrac{a^3}{x}}$. *Ans.* $\sqrt[4]{\dfrac{a^5}{b^2x}}$.

DIVISION OF RADICALS.

CASE I.

207. To divide radicals of the same degree.

Since the root of a fraction, or of the quotient of one quantity divided by another, is obtained by extracting the root of each term separately (**185**), we have

$$\left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}};$$

Or by the radical sign, $\sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$.

Conversely, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$; hence

The quotient of the roots of two quantities is equal to the root of their quotient.

1 Divide $6a^2\sqrt{xy}$ by $2a\sqrt{x}$.

OPERATION.

$$\frac{6a^2\sqrt{xy}}{2a\sqrt{x}} = 3a\sqrt{y}.$$

ANALYSIS. Dividing the coefficients, we have $3a$ for the new coefficient. And, by the principle stated above, $\sqrt{xy}$ divided by $\sqrt{x} = \sqrt{\frac{xy}{x}} = \sqrt{y}$; the entire quotient, therefore, is $3a\sqrt{y}$. Hence the following

RULE. I. *Divide the coefficient of the dividend by the coefficient of the divisor.*

II. *Divide the quantity in the radical part of the dividend by the quantity in the radical part of the divisor, and place the quotient under the common radical sign.*

III. *Prefix the former quotient to the latter, and reduce the result to its simplest form.*

EXAMPLES FOR PRACTICE.

2. Divide $4\sqrt{abc}$ by $2\sqrt{ac}$. *Ans.* $2\sqrt{b}$.
3. Divide $\sqrt{125a^5xy}$ by $\sqrt{5a^3x}$. *Ans.* $5a\sqrt{y}$.
4. Divide $2\sqrt{200m^4}$ by $\sqrt{2m}$. *Ans.* $20m\sqrt{m}$.
5. Divide $\sqrt{160}$ by $\sqrt{8}$. *Ans.* $2\sqrt{5}$.
6. Divide $\sqrt{54}$ by $\sqrt{6}$. *Ans.* 3.
7. Divide $8\sqrt{72}$ by $2\sqrt{6}$. *Ans.* $8\sqrt{3}$.
8. Divide $3\sqrt{10}$ by $\sqrt{15}$. *Ans.* $\sqrt{6}$.
9. Divide $(a^4b^2c)^{\frac{1}{3}}$ by $(ab)^{\frac{1}{3}}$. *Ans.* $a(bc)^{\frac{1}{3}}$.
10. Divide $12(x^5y^7)^{\frac{1}{4}}$ by $3(xy)^{\frac{1}{4}}$. *Ans.* $4xy(y^2)^{\frac{1}{4}}$.
11. Divide $(a^2b^3cd^4)^{\frac{1}{3}}$ by $(a^2bd)^{\frac{1}{3}}$. *Ans.* $d(b^2c)^{\frac{1}{3}}$.
12. Divide $\sqrt{a^3 - a^4b}$ by $\sqrt{a}$. *Ans.* $a\sqrt{1 - ab}$.
13. Divide $\sqrt[3]{a^2 - b^2}$ by $\sqrt[3]{a + b}$. *Ans.* $\sqrt[3]{a - b}$.
14. Divide $(a^2 - b^2)^{\frac{1}{3}}$ by $(a - b)^{\frac{1}{3}}$. *Ans.* $(a + b)^{\frac{1}{3}}$.
15. Divide $\sqrt{\dfrac{x}{y}}$ by $\sqrt{\dfrac{a}{b}}$. *Ans.* $\dfrac{1}{ay}\sqrt{abxy}$.

CASE II.

208. To divide radicals of different degrees.

1. What is the quotient of $(a^3b^2)^{\frac{1}{6}}$ divided by $a^{\frac{1}{2}}$?

OPERATION.

$$Q = \frac{(a^3b^2)^{\frac{1}{6}}}{a^{\frac{1}{2}}} \quad (1)$$

$$Q^6 = \frac{a^3b^2}{a^3} \quad (2)$$

$$Q^6 = b^2 \quad (3)$$

$$Q = b^{\frac{2}{6}} = b^{\frac{1}{3}} \quad (4)$$

ANALYSIS. Letting Q represent the quotient, we have equation (1). Raising both members to the sixth power, we have (2) an equation without a radical index. Dividing a^3b^2 by a^3 we have (3); and extracting the sixth root and reducing the index of b to its lowest terms we have, (4) the required result.

SECOND OPERATION.

$$a^{\frac{1}{2}} = a^{\frac{3}{6}} = (a^3)^{\frac{1}{6}}$$

$$(a^3b^2)^{\frac{1}{6}} \div (a^3)^{\frac{1}{6}} = b^{\frac{2}{6}} = b^{\frac{1}{3}}$$

ANALYSIS. We first reduce the dividend and divisor to the same radical index, by (**202**), and then dividing as in Case I, we have the same result as before. Hence,

RULE. I. *Reduce the radical parts of the dividend and divisor to a common radical index.*

II. *Divide the rational and radical parts separately, as in Case I.*

EXAMPLES FOR PRACTICE.

2. Divide $(ax^2)^{\frac{1}{3}}$ by $(xy)^{\frac{1}{7}}$. *Ans.* $\sqrt[21]{\frac{a^7x^{11}}{y^3}}$.

3. Divide $\sqrt[5]{x}$ by $\sqrt[3]{x}$. *Ans.* $x^{-\frac{2}{15}}$.

4. Divide 30 by $\sqrt{5}$. *Ans.* $6\sqrt{5}$.

5. Divide $10x(a+c)^{\frac{2}{3}}$ by $5(a+c)^{\frac{1}{4}}$. *Ans.* $2x(a+c)^{\frac{5}{12}}$.

6. Divide $(a^2 - x^2)(m+y)^{\frac{1}{n}}$ by $(a+x)(m+y)^{\frac{n}{m}}$.

Ans. $(a-x)(m+y)^{\frac{m-n^2}{mn}}$.

7. Divide $\sqrt{\frac{2}{5}}$ by $\sqrt[3]{\frac{3}{8}}$. *Ans.* $2\sqrt[6]{\frac{8}{1125}}$.

8. Divide $\frac{\sqrt{ax^3}}{\sqrt[3]{ax}}$ by $\frac{\sqrt[3]{a^2x}}{\sqrt{x}}$. *Ans.* $x\frac{\sqrt[3]{x}}{\sqrt{a}}$.

PRINCIPLES

RELATING TO THE APPLICATION OF INVOLUTION AND EVOLUTION.

209. The application of *Involution* to the solution of radical equations, is governed by the principles illustrated by the three following examples.

1. Raise $\sqrt{a}$ to the second power.

OPERATION.

$$\sqrt{a} \times \sqrt{a} = \sqrt{a^2} = a$$

ANALYSIS. Placing the product of the letters under the common radical sign, (**205**), we have $\sqrt{a^2} = a$.

2. Raise $\sqrt{a} + b$ to the second power.

OPERATION.

$$(\sqrt{a} + b)^2 = a + 2b\sqrt{a} + b^2$$

ANALYSIS. Since the given quantity is a binomial, we write a, the square of the first term; $2b\sqrt{a}$, twice the product of the two terms; and b^2, the square of the second term.

3. Raise $\sqrt{a} + \sqrt{b}$ to the second power.

OPERATION.

$$(\sqrt{a} + \sqrt{b})^2 = a + 2\sqrt{ab} + b$$

ANALYSIS. Since the quantity is a binomial, we write a, the square of the first term; $2\sqrt{ab}$, twice the product of the two terms; and b, the square of the second term.

These three examples establish the following principles:

I. *If a radical quantity be involved to a power corresponding to the radical index, the radical sign will be removed.* (1.)

II. *If a quantity containing both radical and rational terms be raised to any power, the radical sign will not be removed.* (2).

III. *If a quantity consisting of two radical terms of the second degree be squared, the result will contain but a single radical term.* (3).

210. The application of *Evolution* to the solution of equations above the first degree, is governed by the principles illustrated in the following examples:

1. Extract the nth root of a^n.

OPERATION.

$$\sqrt[n]{a^n} = a$$

ANALYSIS. We divide the exponent of the power of the given quantity, by the index of the required root (**185**), and obtain 1 for the exponent of the root, which is omitted in the written result.

2. Extract the nth root of $a^n + b$.

Analysis. By the principles of involution, every power of a monomial consists of one term only; and the powers of a binomial consist of at least three terms. And since $a^n + b$, the given quantity, has more terms than any power of a monomial, and a less number of terms than any power of a binomial, it cannot be a perfect power, and we therefore indicate its root, thus, $\sqrt[n]{a^n + b}$.

3. Extract the square root of $a^4b^2 + 2a^3b^2 + a^2b^2$.

OPERATION.

$$\sqrt{a^4b^2} = a^2b \qquad (1)$$
$$\sqrt{a^2b^2} = ab \qquad (2)$$
$$2 \times a^2b \times ab = 2a^3b^2 \qquad (3)$$
$$\sqrt{a^4b^2 + 2a^3b^2 + a^2b^2} = a^2b + ab$$

Analysis. We find by trial, (1) and (2), that two of the given terms are perfect squares, and that twice the product of their square roots is equal to the other term of the given quantity (3). This answers the condition of a binomial square (**186**), and we have $a^2b + ab$ for the required root.

The principles illustrated by these three examples may be stated as follows:

I. *The exponent of a quantity will be removed by extracting the root whose index corresponds to the exponent.* (1).

II. *The root of a binomial is necessarily a surd, and a binomial always becomes a radical by evolution.* (2).

III. *A trinomial is a perfect square when two of its terms are perfect squares and positive, and the remaining term is twice the product of the square roots of the others, and either positive or negative.* (3).

SIMPLE EQUATIONS

CONTAINING RADICAL QUANTITIES.

211. 1. Given $4 + \sqrt{x - 3} = 7$ to find the value of x.

OPERATION.

$$4 + \sqrt{x - 3} = 7 \qquad (1)$$
$$\sqrt{x - 3} = 3 \qquad (2)$$
$$x - 3 = 9 \qquad (3)$$
$$x = 12 \qquad (4)$$

Analysis. We first transpose 4 to the second member, so that the radical may stand alone, and obtain (2). We next square both members to clear the equation of the radical sign (**209**, I), and obtain (3). Reducing, we have $x = 12$.

2. Given $\sqrt{x-2} + \sqrt{x+6} = 4$ to find x.

OPERATION.

$$\sqrt{x-2}+\sqrt{x+6}=4 \qquad (1)$$
$$\sqrt{x-2}=4-\sqrt{x+6} \qquad (2)$$
$$x-2=16-8\sqrt{x+6}+x+6 \qquad (3)$$
$$\sqrt{x+6}=3 \qquad (4)$$
$$x+6=9 \qquad (5)$$
$$x=3 \qquad (6)$$

ANALYSIS. In order to avoid the involution of x to the second power, we transpose one of the radicals to the second member, and obtain (2). Involving both members to the second power, we have (3), an equation containing only one radical. Transposing and reducing, we have (4), in which the radical stands alone. Squaring both members, we obtain (5), and reducing, $x = 3$.

From these examples, we derive the following

RULE. I. *If the equation have but one radical term containing the unknown quantity, transpose the terms so as to make the radical stand alone, as one member; then clear the equation of the radical sign by involution, and reduce as usual.*

II. *If there be two or more radical terms, clear the equation of radicals by successive involutions, adjusting the terms at each step according to the principles enunciated in* (**209**).

NOTE. 1. Equations which appear to be higher than the first degree, sometimes become simple equations by reduction of terms.

EXAMPLES FOR PRACTICE.

3. Given $\sqrt{x} + 5 = 9$, to find x. *Ans.* $x = 16$.

4. Given $\sqrt{x-3} = \frac{4}{\sqrt{x-3}}$, to find x. *Ans.* $x = 7$.

5. Given $\sqrt{4 + (x-2)^{\frac{1}{2}}} = 3$, to find x. *Ans.* $x = 27$.

6. Given $x - \sqrt{x^2+6} = -2$, to find x. *Ans.* $x = \frac{1}{2}$.

7. Given $x + \sqrt{x^2-7} = 7$, to find x. *Ans.* $x = 4$.

8. Given $\sqrt{x+12} = 2 + \sqrt{x}$, to find x. *Ans.* $x = 4$.

9 Given $2 + (3x)^{\frac{1}{3}} = \sqrt{5x + 4}$, to find x.

Ans. $x = 12$.

10. Given $x + 2 = \sqrt{4 + x\sqrt{64 + x^2}}$, to find x.

Ans. $x = 6$.

11. Given $x - \frac{1}{2}\sqrt{x} = \sqrt{x^2 - x}$, to find x.

Ans. $x = \frac{25}{16}$.

12. Given $\sqrt{x - 32} = 16 - \sqrt{x}$, to find x.

NOTE. 2. For brevity, put $a = 16$, and restore the value of a in the final, or reduced equation.

Ans. $x = 81$.

13. Given $\sqrt{3 + x} = \frac{6}{\sqrt{3 + x}}$, to find x. *Ans.* $x = 3$.

14. Given $\sqrt{x - 16} = 8 - \sqrt{x}$, to find x.

Ans. $x = 25$.

15. Given $\sqrt{x + 3a} = 2\sqrt{a}$, to find x. *Ans.* $x = a$.

16. Given $\sqrt{cx + a} = \frac{2a}{\sqrt{2a}}$, to find x. *Ans.* $x = \frac{a}{c}$.

17. Given $\sqrt{x + 2a} = \sqrt{2a} + \sqrt{x - 2a}$, to find x.

Ans. $x = \frac{5a}{2}$.

18. Given $\frac{1}{\sqrt{m^2 - 1}} = \frac{m^2}{\sqrt{m^2 - 1}} - x$, to find x.

Ans. $x = \sqrt{m^2 - 1}$.

19. Given $\frac{\sqrt{x} + 28}{\sqrt{x} + 4} = \frac{\sqrt{x} + 38}{\sqrt{x} + 6}$, to find x.

NOTE.—Place $\sqrt{x} = y$, then find y

Ans. $x = 4$.

20. Given $\sqrt{x} + \sqrt{a + x} = \frac{2a}{(a + x)^{\frac{1}{2}}}$.

Ans. $x = \frac{a}{3}$.

SECTION IV.

QUADRATIC EQUATIONS.

212. A **Quadratic Equation** is an equation of the second degree, or one which contains the second power of the unknown quantity; as $x^2 = 9$, or $x^2 + 3x = a$. Quadratic equations are divided into two classes, *pure* and *affected*.

213. A **Pure Quadratic Equation** is one which contains the *second* power only, of the unknown quantity; as, $x^2 = 25$, or $x^2 + 3ab = 2c$.

NOTE.—A *pure* equation, in general, is an equation which contains only *one power* of the unknown quantity.

214. An **Affected Quadratic Equation** is one which contains both the *second* and the *first* powers of the unknown quantity; as, $x^2 + 3x = 10$.

215. The **Root** of an equation is such a value as, when substituted for the unknown quantity, will satisfy the equation.

PURE QUADRATICS.

216. Since a pure quadratic equation contains only the second power of the unknown quantity, the unknown terms may always be united into one by making the unknown quantity the unit of addition. Hence,

Every pure equation of the second degree can be reduced to the form of $ax^2 = b$; in which a and b are supposed to represent any quantities whatever. Thus the equation

$$3x^2 - 21 = 7 - x^2.$$

becomes, by transposing and uniting terms,

$$4x^2 = 28.$$

And representing 4 by a, and 28 by b, we have

$$ax^2 = b \quad \text{(A)}$$

1. Given $2x^2 = \frac{x^2}{4} + 7$, to find the value of x.

OPERATION.

$$2x^2 = \frac{x^2}{4} + 7 \quad (1)$$
$$8x^2 = x^2 + 28 \quad (2)$$
$$7x^2 = 28 \quad (3)$$
$$x^2 = 4 \quad (4)$$
$$x = \pm 2 \quad (5)$$

VERIFICATION.

$$(+2)^2 = 4 \quad (1)$$
$$(-2)^2 = 4 \quad (2)$$
$$\text{Or, } 2 \times 2^2 = \frac{2^2}{4} + 7 \quad (1)$$
$$2 \times (-2)^2 = \frac{(-2)^2}{4} + 7 \quad (2)$$

ANALYSIS. Clearing the equation of fractions, we obtain (2). Transposing and uniting terms containing x^2, we have (3), an equation in the form of (A). Dividing both members, we obtain (4). To remove the exponent of x^2, we take the square root of both members (**210**, I), which does not destroy their equality (Ax. 9), and obtain (5), in which x has two values, $+2$, and -2. Substituting these values successively in (4), or in (1), each value satisfies the equation, as seen in the verification. Hence,

Every pure quadratic equation has two roots, equal in numerical value, but of opposite signs.

2. Given $mx^2 + 2d = cx^2 + n$ to find x.

OPERATION.

$$mx^2 + 2d = cx^2 + n \quad (1)$$
$$mx^2 - cx^2 = n - 2d \quad (2)$$
$$(m - c)x^2 = n - 2d \quad (3)$$
$$x^2 = \frac{n - 2d}{m - c} \quad (4)$$
$$x = \pm\sqrt{\frac{n - 2d}{m - c}} \quad (5)$$

ANALYSIS. Transposing, we have (2). Factoring with reference to x^2, we obtain (3), which is in the form of (A). Dividing by $m - c$, we have (4); and extracting the square root of both members, we have (5).

From these illustrations we deduce the following

RULE. I. *Reduce the equation to the form of $ax^2 = b$.*

II. *Divide by the coefficient of the unknown quantity, and extract the square root of both members of the resulting equation.*

EXAMPLES FOR PRACTICE.

3. Given $x^2 - 16 = 20$, to find x. *Ans.* $x = \pm 6$.

4. Given $(3x^2 - 14)2 = 2x^2 + 8$, to find x.
Ans. $x = \pm 3$.

5. Given $\frac{3x^2}{8} + 8 = x^2 - 2$, to find x. *Ans.* $x = \pm 4$.

6. Given $x^2 + 1 = \frac{x^2}{4} + 4$, to find x. *Ans.* $x = \pm 2$.

7. Given $\frac{3x^2 + 5}{10} - \frac{2x^2 - 5}{10} = 1$, to find x.
Ans. $x = 0$.

8. Given $\frac{x(9 + 2x)}{15} = \frac{3x + 6}{5}$, to find x.
Ans. $x = \pm 3$.

9. Given $\frac{6x^2 - 10}{4} = x^2 - 1$, to find x.
Ans. $x = \pm 1.73205+$.

10. Given $3x^2 - 29 = \frac{x^2}{4} + 510$, to find x.
Ans. $x = \pm 14$.

11. Given $cx^2 - a^2c = \frac{a^2x^2}{c}$, to find x.
Ans. $x = \pm \frac{ac}{\sqrt{c^2 - a^2}}$.

12. Given $\frac{ax^2(a - 2)}{1 + x} = 1 - x$, to find x.
Ans. $x = \pm \frac{1}{a - 1}$.

NOTE.—Equations containing radical quantities often become quadratic when cleared of the radical sign.

13. Given $\sqrt{x^2 - 5} = \frac{2x}{3}$, to find x. *Ans.* $x = \pm 3$.

14. Given $\sqrt{\frac{20x^2 - 9}{4x}} = \sqrt{x}$, to find x.
Ans. $x = \pm \frac{3}{4}$

15. Given $18 - \sqrt{x^2 - 44} = 8$, to find x.
Ans. $x = \pm 12$.

16. Given $\sqrt{x-3} = \dfrac{2\sqrt{10}}{\sqrt{x+3}}$, to find x.

Ans. $x = \pm 7$.

17. Given $x\sqrt{a^2+x^2} = a^2 - x^2$, to find x.

Ans. $x = \pm a\sqrt{\frac{1}{3}}$.

18. Given $\sqrt{x^2-a^2} = a\sqrt{m-1}$, to find x.

Ans. $x = \pm a\sqrt{m}$.

PROBLEMS

PRODUCING PURE QUADRATIC EQUATIONS.

217. A problem may often furnish either a pure or an affected quadratic equation, according to the notation assumed.

1. Find two numbers whose difference is 6, and whose product is 40.

SOLUTION.

Let $x - 3$ = the less number;
$x + 3$ = the greater.

$$x^2 - 9 = 40 \quad (1)$$
$$x^2 = 49 \quad (2)$$
$$x = 7 \quad (3)$$

$x - 3 = 4$, the less number;
$x + 3 = 10$, the greater.

ANALYSIS. We represent the less number by $x-3$, and the greater by $x+3$, thus making the difference of the numbers 6, according to the first condition of the problem. Multiplying $x-3$ by $x+3$, we obtain $x^2 - 9$, the product of the two numbers, which we put equal to 40, according to the second condition of the problem, and we obtain (1), a pure equation. Reducing, we find $x = 7$, if we use only the plus sign. Then $x - 3 = 4$, the less; and $x + 3 = 10$, the greater.

NOTE.—If we take $x = -7$, in the above problem, we shall have $x - 3 = -10$, the less number, and $x + 3 = -4$. the greater. This result, considered in an *algebraic* sense, satisfies the conditions of the problem, for $-4-(-10) = 6$, the difference, and $(-4) \times (-10) = 40$, the product. In general, however, such a value will be taken for the unknown quantity, as will satisfy the conditions *arithmetically*.

This problem will give rise to an affected quadratic, if we assume the following notation:

$$\begin{aligned} &\text{Let} & x &= \text{less number,} \\ & & x + 6 &= \text{greater,} \\ &\text{Then} & x^2 + 6x &= 40, \end{aligned}$$

an equation containing both powers of x, and for the solution of which rules will be given hereafter. In the problems which follow, an affected quadratic may be avoided by proper notation.

2. The sum of two numbers is 6, and the sum of their cubes is 72; what are the numbers?

SOLUTION.

$$\begin{aligned} \text{Let} \quad 2x &= \text{difference;} \\ 3 + x &= \text{greater number;} \\ 3 - x &= \text{less.} \\ (3 + x)^3 &= 27 + 27x + 9x^2 + x^3, \text{ cube of greater;} \\ (3 - x)^3 &= 27 - 27x + 9x^2 - x^3, \text{ cube of less.} \\ \hline 54 &+ 18x^2 = 72 \qquad (1) \\ x^2 &= 1 \qquad (2) \\ x &= 1 \qquad (3) \\ 3 + x &= 4, \text{ greater number;} \\ 3 - x &= 2, \text{ less.} \end{aligned}$$

ANALYSIS. We let $2x$ represent the difference of the two numbers, $3+x$, half the sum plus half the difference, the greater (**153**); and $3-x$, half the sum minus half the difference, the less; and as the sum of these quantities is 6, this notation satisfies the first condition of the problem. Cubing each, and adding the results, we obtain for the sum of the cubes, $54 + 18x^2$, which we put equal to 72, according to the second condition of the problem. Reducing, we obtain $3 + x = 4$, the greater number; and $3 - x = 2$, the less.

3. A and B distributed 1200 dollars each among a certain number of persons. A relieved 40 persons more than B, and B gave to each individual 5 dollars more than A; how many were relieved by A and B?

SOLUTION.

Let $x + 20 =$ the number relieved by A;
$x - 20 =$ the number relieved by B.

Then $\dfrac{1200}{x+20} + 5 = \dfrac{1200}{x-20}$ (1)

Dividing (1) by 5 $\dfrac{240}{x+20} + 1 = \dfrac{240}{x-20}$ (2)

Assume $a = 20$ and $b = 240$

Eq. (2) becomes $\dfrac{b}{x+a} + 1 = \dfrac{b}{x-a}$ (3)

Reducing (3) $x^2 = a(2b + a)$ (4)

Restoring values of a and b, $x^2 = 10000$ (5)

By evolution $x = 100$ (6)

Hence, $\begin{cases} x + 20 = 120, \text{ number A relieved;} \\ x - 20 = 80, \quad \text{" B "} \end{cases}$

4. Divide the number 56 into two such parts, that their product shall be 640. *Ans.* 40 and 16.

5. Find a number, such that one third of it multiplied by one fourth of it, shall produce 108. *Ans.* 36.

6. What number is that, whose square plus 18 is equal to half its square plus $30\frac{1}{2}$? *Ans.* 5.

7. What two numbers are those, which are to each other as 5 to 6, and the difference of whose squares is 44?

NOTE. — Let $6x =$ the greater, and $5x =$ the less.

Ans. 10 and 12.

8. What two numbers are those, which are to each other as 3 to 4, and the difference of whose squares is 28?

Ans. 6 and 8.

9. What two numbers are those, whose product is 144, and the quotient of the greater divided by the less is 16?

Ans. 48 and 8.

10. The length of a certain lot of land is to its breadth as 9 to 5, and its contents are 405 square feet. Required the length and breadth in feet. *Ans.* 27 and 15.

11. What two numbers are those whose difference is to the greater as 2 to 9, and the difference of whose squares is 128?
Ans. 18 and 14.

12. Find two numbers in the proportion of $\frac{1}{2}$ to $\frac{2}{3}$, the sum of whose squares shall be 225?

NOTE. — Multiplying the fractions $\frac{1}{2}$ and $\frac{2}{3}$ by 6, or reducing them to a common denominator, we find their ratio to be 3 to 4.

Ans. 9 and 12.

13. There is a rectangular field whose breadth is $\frac{5}{6}$ of the length. After laying out $\frac{1}{6}$ of the whole ground for a garden, it was found that there were left 625 square rods for mowing. Required the length and breadth of the field.
Ans. Length, 30 rods; breadth, 25.

14. Two men talking of their ages, one said that he was 94 years old. "Then," replied the younger, "the sum of your age and mine, multiplied by the difference between our ages, will produce 8512." What was the age of the younger?
Ans. 18 years.

15. A fisherman being asked how many fish he had caught, replied, "If you add 11 to the square of the number, 9 times the square root of the sum, diminished by 4, will equal 50." How many had he caught? *Ans.* 5.

16. A merchant gains in trade a sum, to which 320 dollars bears the same proportion as five times the sum does to 2500 dollars; what is the sum? *Ans.* $400.

17. What number is that, the fourth part of whose square being subtracted from 8, leaves a remainder of 4?
Ans. 4.

18. There is a stack of hay, whose length, breadth and height are to each other as the numbers 5, 4 and 3. It is worth as

many cents per cubic foot as it is feet in breadth; and the whole worth, at that rate, is 192 times as many cents as there are square feet in the bottom of the stack. Required the dimensions of the stack.

SOLUTION.

$$\begin{aligned}
\text{Let } 5x &= \text{length};\\
4x &= \text{breadth};\\
3x &= \text{height};\\
5x \times 4x \times 3x &= \text{cubic feet in stack};\\
5x \times 4x &= \text{square feet in bottom};\\
5x \times 4x \times 3x \times 4x &= \text{cost};\\
192 \times 5x \times 4x &= \text{cost}.
\end{aligned}$$

$$\begin{aligned}
5x \times 4x \times 3x \times 4x &= 192 \times 5x \times 4x \quad (1)\\
3x \times 4x &= 192 \quad (2)\\
x^2 &= 16 \quad (3)\\
x &= 4 \quad (4)
\end{aligned}$$

Ans. Length, 20 feet; breadth, 16; height, 12.

ANALYSIS. According to the given proportions, we let $5x$, $4x$, and $3x$ represent the three dimensions. Then $5x \times 4x \times 3x$, their indicated product, will be the solid contents; and $5x \times 4x$, the area. According to the conditions of the problem, we multiply the cubic contents by $4x$, the breadth, and the square contents of the bottom by 192, and obtain two values for the cost, which being put equal to each other, give (1). Canceling the factors $5x$ and $4x$ from both members, we have (2). Again canceling 3×4, or 12, from both members, we have (3), which gives $x = 4$. Hence, $5x = 20$ feet, $4x = 16$, and $3x = 12$, the required dimensions.

NOTE.—The advantage of keeping the factors separate, as in the solution just given, has been fully illustrated in the former part of the book. The pupil may apply the same method to some of the examples which follow.

19. A man purchased a field, the length of which was to its breadth as 8 to 5. The number of dollars paid per acre was equal to the number of rods in the length of the field: and the number of dollars given for the whole, was equal to 13

times the number of rods round the field. Required the length and breadth of the field.

Ans. Length, 104 rods; breadth, 65.

20. There are three numbers in the proportion of 2, 3, and 5; and their product is equal to 108 times their sum. Required the numbers. *Ans.* 12, 18, and 30.

21. It is required to divide the number of 14 into two such parts, that the quotient of the greater divided by the less, may be to the quotient of the less divided by the greater, as 16 : 9.

Ans. The parts are 8 and 6.

22. What two numbers are those whose sum is 12, and whose product is 35? *Ans.* 7 and 5

NOTE. — For notation, see 2d problem.

23. The difference of two numbers is 6, and the sum of their squares is 50; what are the numbers? *Ans.* 7 and 1.

24. The difference of two numbers is 8, and their product is 240; what are the numbers? *Ans.* 12 and 20.

AFFECTED QUADRATICS.

218. Since an affected quadratic equation contains both the first and second powers of the unknown quantity, the equation will contain two unknown terms, and only two, after the coefficients of each power are united.

Thus the equation,

$$3x^2 - 12x = 180 - x^2 + 4x,$$

becomes, by transposition, and reduction of terms,

$$4x^2 - 16x = 180,$$

Or,
$$x^2 - 4x = 45.$$

And if we represent -4 by $2a$, and 45 by b, we have

$$x^2 + 2ax = b \quad \text{(A)}.$$ Hence,

Every affected quadratic equation can be reduced to the form of $x^2 + 2ax = b$, in which $2a$ and b are supposed to represent any quantities whatever, positive or negative.

219. Since the first member of the general equation (A), is a binomial, its root is a surd (**210**, II), and the equation in that form cannot be reduced by evolution. We observe, however, that x^2, the first term of (A), is a perfect square, and $2ax$, the second term, contains x, the root of this square; and it only requires that another square be added, such that twice the product of the *two roots* shall be equal to $2ax$, the second term, to constitute this member a perfect square (**210**, III). The square to be added will evidently be a^2, giving

$$x^2 + 2ax + a^2 = b + a^2 \quad \text{(B)},$$

in which the first member is a perfect square.

But a, whose square, (a^2), we have added, is half the coefficient of x in the second term. Hence,

If the square of half the coefficient of the first power of the unknown quantity be added to both members of a quadratic equation in the form of $x^2 + 2ax = b$, the first member will become a perfect square.

NOTE.—The term, a^2, is added to the first member to *complete the square;* and to the second member to preserve the equality. (Ax. 1.)

EXAMPLES FOR PRACTICE.

Complete the square in each of the following equations:

1. $x^2 + 4x = 96$. *Ans.* $x^2 + 4x + 4 = 96 + 4$.
2. $x^2 - 4x = 45$. *Ans.* $x^2 - 4x + 4 = 49$.
3. $x^2 - 7x = 8$. *Ans.* $x^2 - 7x + \frac{49}{4} = \frac{81}{4}$.
4. $x^2 + 2x = 15$. *Ans.* $x^2 + 2x + 1 = 16$.
5. $x^2 + 12x = 28$. *Ans.* $x^2 + 12x + 36 = 64$.
6. $x^2 + 6x = 16$. *Ans.* $x^2 + 6x + 9 = 25$.
7. $x^2 - 15x = -54$. *Ans.* $x^2 - 15x + \frac{225}{4} = \frac{9}{4}$.
8. $x^2 - \frac{2}{3}x = \frac{13}{3}$. *Ans.* $x^2 - \frac{2}{3}x + \frac{1}{9} = \frac{40}{9}$.

9. $x^2 - \frac{5}{6}x = \frac{1}{6}$. *Ans.* $x^2 - \frac{5x}{6} + \frac{25}{144} = \frac{49}{144}$

10. $x^2 - \frac{a}{b}x = \frac{c}{d}$. *Ans.* $x^2 - \frac{a}{b}x + \frac{a^2}{4b^2} = \frac{c}{d} + \frac{a^2}{4b^2}$

220. Reducing the equation,

$$x^2 + 2ax + a^2 = b + a^2.$$

By evolution, we have $x + a = \pm\sqrt{b+a^2}$.

Transposing a, $x = -a + \sqrt{b+a^2}$, 1st root.

Or, $x = -a - \sqrt{b+a^2}$, 2d root.

Hence,

Every affected quadratic equation has two roots, unequal in numerical value.

1. Given $x^2 - \frac{4 - 32x}{3} = \frac{8(1+x)}{3} + 16$, to find the values of x.

OPERATION.

$$x^2 - \frac{4 - 32x}{3} = \frac{8(1+x)}{3} + 16 \quad (1)$$
$$3x^2 - 4 + 32x = 8 + 8x + 48 \quad (2)$$
$$3x^2 + 24x = 60 \quad (3)$$
$$x^2 + 8x = 20 \quad (4)$$
$$x^2 + 8x + 16 = 36 \quad (5)$$
$$x + 4 = \pm 6 \quad (6)$$
$$x = 2 \quad (7)$$
Or, $$x = -10 \quad (8)$$

ANALYSIS. Clearing of fractions, we obtain (2). Transposing and uniting, we have (3). Dividing by 3, we have (4), an equation in the form of (A). As 8 is the coefficient of x, we add 16, the square of one half of 8, to both members, and obtain (5), in which the first member is a perfect square. Extracting the square root of both members, we have (6). Transposing 4, and uniting -4 with $+6$, the plus value of the root, we have $x = 2$, the first value of x; uniting the -4 with

-6, the minus value of the root, we have $x = -10$, the second value of x. Hence, for the solution of an affected quadratic equation, we have the following general

Rule. I. *Reduce the given equation to the form of* $x^2 + 2ax = b$.

II. *Complete the square of the first member, by adding to both members of the equation the square of one half the coefficient of the second term.*

III. *Extract the square root of both members, and reduce the resulting equation.*

EXAMPLES FOR PRACTICE.

Reduce the following equations :

2. $x^2 + 4x = 96$. *Ans.* $x = 8$, or -12.
3. $x^2 - 4x = 45$. *Ans.* $x = 9$, or -5.
4. $x^2 - 7x = 8$. *Ans.* $x = 8$, or -1.
5. $x^2 + 2x = 15$. *Ans.* $x = 3$, or -5.
6. $x^2 + 12x = 28$. *Ans.* $x = 2$, or -14.
7. $x^2 + 6x = 16$. *Ans.* $x = 2$, or -8.
8. $x^2 - 15x = -54$. *Ans.* $x = 9$, or 6.
9. $x^2 - \frac{2}{3}x = \frac{133}{3}$. *Ans.* $x = 7$, or $-\frac{19}{3}$.
10. $x^2 - \frac{5}{6}x = \frac{1}{6}$. *Ans.* $x = 1$, or $-\frac{1}{6}$.
11. $x^2 - \frac{a}{b}x = \frac{c}{d}$. *Ans.* $x = \frac{a}{2b} \pm \left(\frac{c}{d} + \frac{a^2}{4b^2}\right)^{\frac{1}{2}}$.

Notes. 1. The ten preceding examples are all in the form of $x^2 + 2ax = b$, and require no application of the first step of the rule.

2. If in the preparation of the equations which follow, the square of the unknown quantity appears with the *minus sign*, make it positive by changing all the signs of the equation.

12. $3x^2 - 25x = -72 + 5x$. *Ans.* $x = 6$, or 4.
13. $2x^2 + 100 = 32x - 10$. *Ans.* $x = 11$, or 5.
14. $6x - 300 = 204 - 3x^2$. *Ans.* $x = 12$, or -14.
15. $5x^2 + 80 = -505 - 110x$. *Ans.* $x = -9$, or -13.
16. $2x^2 - 9x = -4$. *Ans.* $x = 4$, or $\frac{1}{2}$.

17. $1 - \frac{x}{2} = 5 - \frac{36}{x + 2}$. *Ans.* $x = 4$, or -14.

18. $3 - x^2 = \frac{2x - 76}{3}$. *Ans.* $x = 5$, or $-5\frac{2}{3}$.

19. $x^2 + \frac{x}{2} = \frac{2x^2}{5} - \frac{x}{5} + \frac{13}{10}$. *Ans.* $x = 1$, or $-2\frac{1}{6}$.

20. $\frac{x^2}{4} - 30 + x = 2x - 22$. *Ans.* $x = 8$, or -4.

21. $\frac{x^2}{2} - \frac{x}{3} + 7\frac{1}{2} = 8\frac{1}{8}$. *Ans.* $x = 1\frac{1}{2}$, or $-\frac{5}{6}$.

22. $\frac{x^2}{4} - 15 = \frac{2x}{3} - 14\frac{3}{4}$. *Ans.* $x = 3$. or $-\frac{1}{3}$.

23. $\frac{x}{x + 8} = \frac{x + 3}{2x + 1}$. *Ans.* $x = 12$, or -2.

24. $x - 1 + \frac{2}{x - 4} = 0$. *Ans.* $x = 3$. or 2.

25. $\frac{22 - x}{20} = \frac{15 - x}{x - 6}$. *Ans.* $x = 36$, or 12.

26. $\frac{2x - 7}{x - 1} = \frac{x + 1}{3x + 3}$. *Ans.* $x = 4$, or -1.

27. $x^2 + 2ax = 3a^2$. *Ans.* $x = a$, or $-3a$.

28. $x^2 - 4cx = 4cd - 2dx - 3c^2 - d^2$.
Ans. $x = 3c - d$, or $c - d$.

29. $x^2 - 2ax = m^2 - a^2$. *Ans.* $x = a \pm m$.

30. $x^2 - 2cx = 4m - c^2$. *Ans.* $x = c \pm 2\sqrt{m}$.

SECOND METHOD OF COMPLETING THE SQUARE.

221. It frequently happens in the reduction of a quadratic to the form of $x^2 + 2ax = b$, that some or all of the terms become fractional, and render the solution complex and difficult. In such cases, it will be sufficient to reduce the three parts of the equation to the simplest entire quantities, by uniting terms, and dividing the equation by the greatest common

divisor of the two members. The equation will then be in the form of

$$ax^2 + bx = c \quad (\text{A}),$$

in which a, b, and c, are entire quantities, prime to each other.

To render the first member of equation (A), a *binomial square*, we may make its first term a perfect square, by multiplying the equation by a, and afterwards *complete* the square by the rule already given. The operation will appear as follows:

$$ax^2 + bx = c; \quad (1)$$

Multiplying (1) by a, $\quad a^2x^2 + bax = ac; \quad (2)$

Putting $\quad y = ax;$

And (2) becomes $\quad y^2 + by = ca; \quad (3)$

Completing the square, $\quad y^2 + by + \frac{b^2}{4} = ca + \frac{b^2}{4}. \quad (4)$

The first member of equation (4) is a binomial square; but one of the terms is fractional, a condition which we are seeking to avoid. The denominator of the fraction is the square number, 4; and if the equation be multiplied by 4 to clear it of fractions, the first member will still be a square, because it will consist of *square factors*. Hence, multiplying the equation by 4, we obtain,

$$4y^2 + 4by + b^2 = 4ca + b^2; \quad (5)$$

Restoring value of y, $\quad 4a^2x^2 + 4abx + b^2 = 4ca + b^2. \quad (6)$

Factoring this result, and comparing it with the primitive equation, thus,

Primitive equation, $\quad ax^2 + bx = c; \quad (1)$

Square completed, $\quad 4a(ax^2 + bx) + b^2 = 4a(c) + b^2. \quad (6)$

we perceive that equation (6) may be obtained by multiplying equation (1) by $4a$, and adding b^2, to both members of the result. Hence,

If a quadratic equation in the form of $ax^2 + bx = c$, be multiplied by 4 times the coefficient of the second power of the unknown quantity, and the square of the coefficient of the first power be added to both sides, the first member will become a perfect square.

1. Given $5x^2 + 4x = 204$, to find the values of x.

OPERATION.

$$5x^2 + 4x = 204 \quad (1)$$
$$100x^2 + 80x + 16 = 4096 \quad (2)$$
$$10x + 4 = \pm 64 \quad (3)$$
$$10x = 60 \text{ or } -68 \quad (4)$$
$$x = 6 \text{ or } -6\tfrac{4}{5} \quad (5)$$

ANALYSIS. To complete the square, we multiply equation (1) by 4 times 5, and add the square of 4 to both members, and obtain (2). Extracting the square root of both members, we have (3). Transposing 4, we have (4); and dividing by 10, we obtain $x = 6$, or $-6\frac{4}{5}$.

2. Given $x^2 - 5x = -6$, to find the values of x.

OPERATION.

$$x^2 - 5x = -6 \quad (1)$$
$$4x^2 - (\ \ ) + 25 = 1 \quad (2)$$
$$2x - 5 = \pm 1 \quad (3)$$
$$x = 3 \text{ or } 2 \quad (4)$$

ANALYSIS. In this example, 4 times the coefficient of x^2 is 4. We therefore multiply by 4, and add the square of 5 to both members, and obtain (2). Reducing by evolution and transposition, we have $x = 3$, or 2. As the second term of a binomial square is dropped in extracting the square root, we may place ( ) in the equation, preceded by the proper sign, when we complete the square.

From these examples we derive the following

RULE. I. *Reduce the equation to the form of $ax^2 + bx = c$, in which the three terms are entire, and prime to each other.*

II. *Multiply the equation by 4 times the coefficient of x^2, and add the square of the coefficient of x to both members.*

III. *Extract the square root of both members, and reduce the resulting equation.*

NOTE.—If the coefficient of x^2 is 1, the second method will be applied with advantage, provided the coefficient of x is *odd;* but if it is *even*, the first rule is preferable.

EXAMPLES FOR PRACTICE.

3. Given $2x^2 - 5x = 117$, to find the values of x.
Ans. $x = 9$, or $-6\frac{1}{2}$.

4. Given $3x^2 - 5x = 28$, to find the values of x.
Ans. $x = 4$, or $-\frac{7}{3}$.

5. Given $3x^2 - x = 70$, to find the values of x.
Ans. $x = 5$, or $-\frac{14}{3}$.

6 Given $5x^2 + 4x = 273$, to find the values of x.
Ans. $x = 7$, or $-7\frac{4}{5}$.

7. Given $2x^2 + 3x = 65$, to find the values of x.
Ans. $x = 5$, or $-6\frac{1}{2}$.

8. Given $3x^2 + 5x = 42$, to find the values of x.
Ans. $x = 3$, or $-4\frac{2}{3}$.

9. Given $8x^2 - 7x + 16 = 181$, to find x.
Ans. $x = 5$, or $-4\frac{1}{8}$.

10. Given $10x^2 - 8x + 8 = 320$, to find x.
Ans. $x = 6$, or $-5\frac{1}{5}$.

11. Given $3x^2 + 2x = 4$, to find x.
Ans. $x = -\frac{1}{3} \pm \frac{1}{3}\sqrt{13}$.

12. Given $5x^2 + 7x = 7$, to find x.
Ans. $x = -\frac{7}{10} \pm \frac{3}{10}\sqrt{21}$.

13. Given $\frac{240}{x} + \frac{4}{10} = \frac{216}{x - 15}$, to find x. *Ans.* $x = \begin{matrix} 75, \text{ or} \\ -120. \end{matrix}$

14. Given $12 + x = \frac{x^2 + 12x}{5}$, to find x.
Ans. $x = 5$, or -12.

Find the approximate roots of the following equations:

15. $x^2 - 5x = -2$.
Ans. $x = 4.5615 +$, or $.4384 +$.

16. $2x^2 - 3x = 12$.
Ans. $x = 3.3117 +$, or $-1.8117 +$.

17. $3x^2 - x = 1$.
Ans. $x = .7675 +$, or $-.4342 +$.

18. $x^2 - x = 1$.

Ans. $x = 1.6180 +$, or $-.6180 +$.

19. $4x^2 + 3x = 5$.

Ans. $x = .8042 +$, or $-1.5542 +$.

20. $x^2 - 7x = -11$.

Ans. $x = 4.6180 +$, or $2.3820 +$.

HIGHER EQUATIONS IN THE QUADRATIC FORM.

222. Any equation is in the *quadratic form*, when it contains but *two powers* of the unknown quantity, and the index of the higher power is *twice the index of the lower.* Such equations are reducible to one of the following forms:

$$x^{2n} + ax^n = b; \text{ or,}$$
$$ax^{2n} + bx^n = c;$$

and may therefore be reduced by one of the rules for quadratics.

1. Given $x^6 - 4x^3 = 621$, to find the values of x.

OPERATION.

$$
\begin{array}{rrll}
\text{Put} & y &= x^3 & \\
& y^2 &= x^6 & \\
\hline
& y^2 - 4y &= 621 & (1) \\
& y^2 - (\ \ ) + 4 &= 625 & (2) \\
& y - 2 &= \pm 25 & (3) \\
& y &= 27 \text{ or } -23 & (4) \\
\text{or,} & x^3 &= 27 \text{ or } -23 & (5) \\
& x &= 3 \text{ or } \sqrt[3]{-23} & (6)
\end{array}
$$

ANALYSIS. To simplify the application of the rule for quadratics, we put $y = x^3$ and $y^2 = x^6$. The given equation then becomes (1), a quadratic in the general form. Solving in the usual manner, we obtain (4). Restoring the value of y, we have (5) a pure equation; and extracting the cube root of both members we have $x = 3$ or $\sqrt[3]{-23}$.

NOTE 1. It will be remembered that the *odd* roots of a negative quantity are *real*, while the *even* roots are *imaginary* (184). Hence, by extracting the cube root of -23, and prefixing the minus sign, we find the approximate value of the second root in the example above. Thus, $\sqrt[3]{-23} = 2.84 +$.

2. Given $x^3 - x^{\frac{3}{2}} = 56$, to find x.

OPERATION.

$$
\begin{array}{llll}
\text{Put} & y = x^{\frac{3}{2}} & & \\
& y^2 = x^3 & & \\
\hline
& y^2 - y = 56 & & (1) \\
& 4y^2 - (\;\;) + 1 = 225 & & (2) \\
& 2y - 1 = \pm 15 & & (3) \\
& y = 8 & \text{or } -7 & (4) \\
\text{or,} & x^{\frac{3}{2}} = 8 & \text{or } -7 & (5) \\
& x^{\frac{1}{2}} = 2 & \text{or } (-7)^{\frac{1}{3}} & (6) \\
& x = 4 & \text{or } (-7)^{\frac{2}{3}} & (7)
\end{array}
$$

ANALYSIS. If we represent $x^{\frac{3}{2}}$ by y, x^3 will be y^2, and the equation, (1), takes the usual quadratic form. Reducing by the second rule, we have (4). Restoring the value of y, we obtain (5). Extracting the cube root of both members, we obtain (6); and squaring both members we have $x=4$ or $(-7)^{\frac{2}{3}}$.

NOTE 2. The expression $(-7)^{\frac{2}{3}}$ signifies the cube root of the second power of -7; or, $\sqrt[3]{49} = 3.65+$.

EXAMPLES FOR PRACTICE.

3. Given $x^4 + 2x^2 = 24$, to find the values of x.

OPERATION.

	$x^4 + 2x^2 = 24$	(1)
Completing square,	$x^4 + (\;\;) + 1 = 25$	(2)
Extracting root,	$x^2 + 1 = \pm 5$	(3)
Transposing,	$x^2 = -1 \pm 5$	(4)
Uniting terms,	$x^2 = 4$, or -6	(5)
Extracting root,	$x = \pm 2$, or $\pm \sqrt{-6}$	(6)

4. Given $x^4 - 3x^2 = 550$, to find the values of x.
Ans. $x = \pm 5$, or $\pm \sqrt{-22}$.

5. Given $3x - x^{\frac{1}{2}} = 44$, to find the values of x.
Ans. $x = 16$, or $13\frac{4}{9}$.

6. Given $x^6 - 7x^3 = 8$, to find the values of x.
Ans. $x = 2$, or -1.

7. Given $x^6 - 6x^3 = 567$, to find the values of x.
Ans. $x = 3$, or $-2.758+$.

POLYNOMIALS UNDER THE QUADRATIC FORM.

223. When a polynomial appears under different powers or fractional exponents, one exponent being twice the other, we may represent the quantity by a single letter, and apply one of the rules for quadratics, as in the last article.

1. Given $(x^2 + 2x)^2 + 2(x^2 + 2x) = 80$, to find the values of x.

OPERATION.

Assume	$y^2 = (x^2 + 2x)^2$;
And	$y = x^2 + 2x$;
Then,	$y^2 + 2y = 80$;
Completing square,	$y^2 + (\ \ ) + 1 = 81$;
By evolution,	$y + 1 = \pm 9$;
Reducing,	$y = 8$, or -10.

Restoring the value of y, we have two equations containing x;

Thus,	$x^2 + 2x = 8$,	or $x^2 + 2x = -10$;
Whence,	$x^2 + 2x + 1 = 9$,	or $x^2 + 2x + 1 = -9$;
And,	$x + 1 = \pm 3$;	or $x + 1 = \pm 3\sqrt{-1}$;
Therefore,	$x = 2$ or -4;	or, $x = 3\sqrt{-1}-1$;
		or, $-(1 + 3\sqrt{-1})$.

EXAMPLES FOR PRACTICE.

2. Given $(x + 3) + 2(x + 3)^{\frac{1}{2}} = 35$, to find the values of x.

OPERATION.

	$(x + 3) + 2(x + 3)^{\frac{1}{2}} = 35$	(1)
Completing square,	$(x + 3) + (\ \ ) + 1 = 36$	(2)
Extracting root,	$\sqrt{x + 3} + 1 = \pm 6$	(3)
Transposing and uniting terms,	$\sqrt{x + 3} = 5$ or -7	(4)
Squaring both members,	$x + 3 = 25$ or 49	(5)
Transposing and uniting,	$x = 22$ or 46	(6)

3. Given $(y^2+2y)^2+4(y^2+2y)=96$, to find one value of y. *Ans.* $y=2$.

4. Given $10+x-(10+x)^{\frac{1}{2}}=12$, to find one value of x. *Ans.* $x=6$

5. Given $\left(\frac{6}{y}+y\right)^2+\left(\frac{6}{y}+y\right)=30$, to find y.

Ans. $y=3$ or 2, or $-3\pm\sqrt{3}$.

6. Given $(x+12)^{\frac{1}{2}}+(x+12)^{\frac{1}{4}}=6$, to find the values of x. *Ans.* $x=4$, or 69.

7. Given $2x^2+3x+9-5\sqrt{2x^2+3x+9}=6$, to find the real values of x.

Ans. $x=3$ or $-4\frac{1}{2}$, or $x=-\frac{3}{4}\pm\frac{1}{4}\sqrt{-55}$.

8. Given $(x+a)^{\frac{1}{2}}+2b(x+a)^{\frac{1}{4}}=3b^2$, to find the values of x. *Ans.* $x=b^4-a$, or $81b^4-a$.

FORMATION OF QUADRATIC EQUATIONS.

224. The **Absolute Term** of an equation is the term or quantity which does not contain the unknown quantity.

225. The roots of quadratics possess certain properties which enable us to reconstruct the equation when its roots are known.

Let us resume the general equation,

$$x^2+2ax=b; \qquad (A)$$

Completing the square, $x^2+2ax+a^2=a^2+b$;

By evolution, $x+a=\pm\sqrt{a^2+b}$;

Hence, $x=-a+\sqrt{a^2+b}$, 1st root;

And, $x=-a-\sqrt{a^2+b}$, 2d root.

Adding these two roots, we have

$$\begin{array}{ll} & -a+\sqrt{a^2+b} \\ & -a-\sqrt{a^2+b} \\ \hline \text{Sum,} & -2a \end{array}$$

Multiplying them together, we have

$$\begin{array}{r} -a+\sqrt{a^2+b} \\ -a-\sqrt{a^2+b} \\ \hline a^2-a\sqrt{a^2+b} \\ +a\sqrt{a^2+b}-(a^2+b) \\ \hline \end{array}$$

Product, $-b.$

If we transpose the absolute term of equation (A) the equation will appear as follows:

$$x^2+2ax-b=0 \qquad \text{(B)}$$

Comparing the *sum* and *product* now obtained, we conclude that in every equation in the form of $x^2+2ax-b=0$,

I. *The sum of the two roots is equal to the coefficient of the unknown quantity in the second term, taken with the contrary sign.*

II. *The product of the two roots is equal to the absolute term taken with its proper sign.*

1. Form the equation whose roots are 4 and -12.

OPERATION.

Algebraic sum of roots,	$4-12=-8$
Product of roots,	$4\times(-12)=-48$
Hence,	$x^2+8x-48=0$

ANALYSIS. The algebraic sum of the roots is -8; and their product is -48. Hence, from PROPERTY (I), we have 8 for the coefficient of the first power of the unknown quantity; and from (II), we have -48 for the absolute term in the first member; hence the equation is $x^2+8x-48=0$.

From these principles and illustrations we have the following

RULE. I. *Write the second power of the unknown quantity for the first term.*

II. *Take the algebraic sum of the roots, with its sign changed, as the coefficient of the unknown quantity in the second term.*

III. *Write the product of the roots with its proper sign for the third term, and place the whole result equal to zero.*

EXAMPLES FOR PRACTICE.

2. Form the equation whose roots are 10 and -7.
Ans. $x^2 - 3x - 70 = 0$.

3. Form the equation whose roots are 12 and -5.
Ans. $x^2 - 7x - 60 = 0$.

4. Form the equation whose roots are 6 and -15.
Ans. $x^2 + 9x - 90 = 0$.

5. Form the equation whose roots are 1 and -2.
Ans. $x^2 + x - 2 = 0$.

6. Form the equation whose roots are 4 and 13.
Ans. $x^2 - 17x + 52 = 0$.

7. Form the equation whose roots are -5 and -3.
Ans. $x^2 + 8x + 15 = 0$.

8 Form the equation whose roots are $4\frac{1}{2}$ and $5\frac{1}{2}$.
Ans. $x^2 - 10x + 24.75 = 0$.

SECOND METHOD.

226. Let us suppose that a and b represent any quantities, and find the product of the binomials $x - a$ and $x - b$; thus,

$$\begin{array}{ll} & x - a \\ & x - b \\ \hline \text{Product,} & x^2 - (a + b)x + ab. \end{array}$$

Now by placing this product equal to zero, thus

$$x^2 - (a + b)x + ab = 0,$$

we form the equation whose roots are a and b; because the coefficient of x, $-(a + b)$, taken with the contrary sign, is

the sum of a and b, (**225**, I); and the absolute term, ab, is the product of a and b (**225**, II). Hence

Every quadratic equation in the form of $x^2 + 2ax - b = 0$ *is composed of two binomial factors, of which the first term in each is the unknown quantity, and the second term, the two roots with their signs changed.*

To illustrate this by a numerical example, take the following equation:

$$x^2 + 4x - 60 = 0 \qquad (1)$$

Transposing, $x^2 + 4x = 60$ (2)

Completing square, $x^2 + (\ ) + 4 = 64$ (3)

By evolution, $x + 2 = \pm 8$ (4)

Whence, $x = 6$, 1st root;

And, $x = -10$, 2d root.

Connecting these roots with x, with their signs changed, and multiplying, we have

$$\begin{array}{l} x - 6 \\ \underline{x + 10} \\ x^2 - 6x \\ \quad\ \ \underline{10x - 60} \\ x^2 + 4x - 60 = 0 \end{array}$$

Thus we have reconstructed equation (1). Hence,

RULE. I. *Connect each root, with its contrary sign, to an unknown quantity.*

II. *Multiply together the binomial factors thus formed, and place the product equal to zero.*

EXAMPLES FOR PRACTICE.

1. Find the equation which has 3 and -2 for its roots. *Ans.* $x^2 - x - 6 = 0$.
2. Find the equation which has 5 and -9 for its roots. *Ans.* $x^2 + 4x - 45 = 0$.
3. Find the equation which has 7 and -7 for its roots. *Ans.* $x^2 - 49 = 0$.

4. Find the equation which has 8 and -12 for its roots.

Ans. $x^2 + 4x - 96 = 0.$

5. Find the equation which has -5 and -7 for its roots.

Ans. $x^2 + 12x + 35 = 0.$

6. Find the equation which has a and $a - b$ for its roots.

Ans. $x^2 - (2a - b)x + a^2 - ab = 0.$

7. Find the equation which has b and $-a$ for its roots.

Ans. $x^2 + (a - b)x - ab = 0.$

FACTORING TRINOMIALS.

227. The principle established in the last article, enables us to resolve any trinomial, in the form of $x^2 + ax + b$, into two binomial factors, either exact or approximate.

1. Resolve $x^2 + 5x + 6$ into two binomial factors.

OPERATION.

$$x^2 + 5x + 6 = -0$$
$$x^2 + 5x = -6$$
$$4x^2 + (\ ) + 25 = 1$$
$$2x + 5 = \pm 1$$
$$x = -3, \text{ or } -2$$
$$(x + 3)\ (x + 2), \text{ factors.}$$

Or, $2 \times 3 = 6$, absolute term,
$2 + 3 = 5$, coefficient of x,
And $(x + 3)\ (x + 2)$, factors.

ANALYSIS. Since the given quantity is in the form of a quadratic equation reduced to one member, we place it equal to zero, and solve the resulting equation. Then, changing the signs of the roots, and connecting them with x, we have $(x + 3)$ $(x + 2)$, the factors that compose the trinomial (**226**). Or, we may find by inspection, two factors of 6, the absolute term, whose sum is equal to 5, the coefficient of x in the middle term, and thus form the binomial factors sought. Hence, the following

RULE. I. *Place the trinomial equal to zero, and reduce the resulting equation.*

II. *Connect each root, with its sign changed, to the lowest power of the literal quantity, and the result will be the binomial factors required.* Or,

Find by inspection two factors of the absolute term, whose sum is equal to the coefficient of the middle term; and connect each with its proper sign, to the literal quantity.

EXAMPLES FOR PRACTICE.

Factor the following trinomials.

2. $x^2 - x - 20$. *Ans.* $(x - 5)(x + 4)$.

3. $a^2 - 7a + 12$. *Ans.* $(a - 3)(a - 4)$.

4. $a^2 - 7a - 8$. *Ans.* $(a - 8)(a + 1)$.

5. $x^2 - x - 30$. *Ans.* $(x - 6)(x + 5)$.

6. $x^2 + 7x - 18$. *Ans.* $(x + 9)(x - 2)$.

7. $x^2 + 11x - 42$. *Ans.* $(x + 14)(x - 3)$.

8. $x^2 + 2x - 5$. *Ans.* $(x - 1.449 +)(x + 3.449 +)$.

9. $x^2 - 8x + 8$. *Ans.* $(x - 6.828+)(x - 1.172 +)$.

THE FOUR FORMS.

228. In the general equation, $x^2 + 2ax = b$, $2a$, as we have seen, may be either positive or negative; and b may be either positive or negative; therefore, for a representation of every variety of quadratic equations, we have the four general forms,

$$x^2 + 2ax = b \qquad (1)$$
$$x^2 - 2ax = b \qquad (2)$$
$$x^2 - 2ax = -b \qquad (3)$$
$$x^2 + 2ax = -b \qquad (4)$$

A reduction of these several equations gives for the values of x as follows:

$$x = -a \pm \sqrt{a^2 + b} \qquad (1)$$
$$x = +a \pm \sqrt{a^2 + b} \qquad (2)$$
$$x = +a \pm \sqrt{a^2 - b} \qquad (3)$$
$$x = -a \pm \sqrt{a^2 - b} \qquad (4)$$

REAL AND IMAGINARY ROOTS.

229. By examining the roots of the four forms given in the last article, we find that in the first and second forms, $a^2 + b$, the quantity under the radical is *positive;* its root can,

therefore, always be taken, either exactly or approximately. But in the third and fourth forms, $a^2 - b$, the quantity under the radical will be *negative* when the term b is greater numerically than a^2; in which case, the root cannot be extracted (**184**), and must be *imaginary*. Hence,

I. *In each of the first and second forms, both roots are always real.*

II. *In each of the third and fourth forms, both roots are imaginary when the absolute term is numerically greater than the square of half the coefficient of the unknown quantity in the second term.*

NOTE.—Imaginary roots of an equation furnished by a problem, indicate that the conditions of the problem are *impossible* or *absurd.*

PROBLEMS

PRODUCING QUADRATIC EQUATIONS.

230. 1. If four times the square of a certain number be diminished by twice the number, it will leave a remainder of 30; what is the number? *Ans.* 3.

NOTE.—The number 3 is the only number that will answer the required conditions; the algebraic expression $-\frac{5}{2}$ will also answer the conditions; but the expression is not a number in an *arithmetical sense.*

2. A person purchased a number of horses for 240 dollars. If he had obtained 3 more for the same money, each horse would have cost him 4 dollars less; required the number of horses. *Ans.* 12.

3. A grazier bought as many sheep as cost him 240 dollars, and after reserving 15 out of the number, he sold the remainder for 216 dollars, and gained 40 cents a head on the number sold; how many sheep did he purchase? *Ans.* 75.

4. A company dining at a house of entertainment, had to pay 3 dollars 50 cents; but before the bill was presented two of them had left, in consequence of which, those who remained, had to pay each 20 cents more than if all had been present; how many persons dined? *Ans.* 7.

5. There is a certain number, which being subtracted from 22, and the remainder multiplied by the number, the product will be 117; what is the number? *Ans.* 13 or 9.

6. In a certain number of hours a man traveled 36 miles; if he had traveled one mile more per hour, it would have taken him 3 hours less to perform his journey; how many miles did he travel per hour? *Ans.* 3 miles.

7. A man being asked how much money he had in his purse, answered, that the square root of the number taken from half the number would give a remainder of 180 dollars; how much money had he? *Ans.* $400.

8. If a certain number be increased by 3, and the square root of the sum be added to the number, the sum will be 17; what is the number? *Ans.* 13.

9. The square of a certain number and 11 times the number make 80; what is the number? *Ans.* 5 or -16.

10. A poulterer going to market to buy turkeys, met with four flocks. In the second flock, were 6 more than 3 times the square root of double the number in the first; the third contained 3 times as many as the first and second; the fourth contained 6 more than the square of one third the number in the third; and the whole number was 1938. How many were in each flock? *Ans.* 18, 24, 126, 1770.

NOTE.—Let $2x^2$ equal the number in the first. Also see (**223**).

11. The plate of a mirror 18 inches by 12, is to be set in a frame of uniform width, and the area of the frame is to be equal to that of the glass; required the width of the frame.

Ans. 3 inches.

12. A square courtyard has a rectangular gravel walk round it. The side of the court is two yards less than six times the width of the gravel walk, and the number of square yards in the walk exceeds the number of yards in the periphery of the court by 164; required the area of the court, exclusive of the walk. *Ans.* 256 yards.

13. A and B start at the same time to travel 150 miles; A travels 3 miles an hour faster than B, and finishes his journey $8\frac{1}{3}$ hours before him; at what rate per hour does each travel?

Ans. 9 and 6 miles per hour.

14. A company at a tavern had 1 dollar 75 cents to pay; but before the bill was paid two of them left, when those who remained had each 10 cents more to pay; how many were in the company at first? *Ans.* 7.

15. A set out from C, toward D, and traveled 7 miles a day. After he had gone 32 miles, B set out from D toward C, and went every day $\frac{1}{19}$ of the whole journey; and after he had traveled as many days as he went miles in a day, he met A; required the distance from C to D.

Ans. 76 or 152 miles; either number will answer the conditions.

QUADRATIC EQUATIONS

CONTAINING TWO UNKNOWN QUANTITIES.

231. In general, two equations essentially quadratic, involving two unknown quantities, depend for their solution on a resulting equation of the fourth degree. A solution may be effected, however, by the rule for quadratics, if the equations come under one of the three following cases:

1st. When one of the equations is simple, and the other quadratic.

2d. When the equations are similar in form, or the unknown quantities are involved and combined in a similar manner.

3d. When the equations are homogeneous.

We give illustrations of the three classes in succession.

1st. Simple and Quadratic.

1. Given $\left\{\begin{array}{l} x + 2y = 9 \\ x^2 + 2y^2 = 33 \end{array}\right\}$, to find x and y.

OPERATION.

	$x + 2y = 9$	(1)
	$x^2 + 2y^2 = 33$	(2)
From (1),	$x = 9 - 2y$	(3)
Squaring (3),	$x^2 = 81 - 36y + 4y^2$	(4)
From (2),	$x^2 = 33 - 2y^2$	(5)
From (4) and (5),	$81 - 36y + 4y^2 = 33 - 2y^2$	(6)
Reducing,	$y^2 - 6y = -8$	(7)
Whence,	$y = 4$ or 2	(8)
And from (1),	$x = 1$ or 5	(9)

2d. SIMILAR EQUATIONS.

2. Given $\left\{ \begin{array}{r} x + y = 10 \\ xy = 24 \end{array} \right\}$, to find x and y,

OPERATION.

	$x + y = 10$	(1)
	$xy = 24$	(2)
Squaring (1),	$x^2 + 2xy + y^2 = 100$	(3)
Multiplying (2) by 4,	$4xy = 96$	(4)
Taking (4) from (3),	$x^2 - 2xy + y^2 = 4$	(5)
Extracting square root of (5),	$x - y = \pm 2$	(6)
But in (1),	$x + y = 10$	(7)
Adding (6) to (7),	$2x = 12$ or 8	(8)
Taking (6) from (7),	$2y = 8$ or 12	(9)
Whence	$x = 6$ or 4	(10)
And	$y = 4$ or 6	(11)

3. Given $\left\{ \begin{array}{r} x + y = 10 \\ x^2 + y^2 = 58 \end{array} \right\}$, to find x and y.

OPERATION.

$$x + y = 10 \quad (1)$$
$$x^2 + y^2 = 58 \quad (2)$$

Squaring (1), $x^2 + 2xy + y^2 = 100$ (3)

Taking (2) from (3), $2xy = 42$ (4)

Taking (4) from (2), $x^2 - 2xy + y^2 = 16$ (5)

Extracting square root of (5), $x - y = \pm 4$ (6)

But in (1), $x + y = 10$ (7)

Whence, $x = 7 \text{ or } 3$ (8)

And $y = 3 \text{ or } 7$ (9)

4. Given $\left\{\begin{array}{l} x + y = 5 \\ x^3 + y^3 = 35 \end{array}\right\}$, to find x and y.

OPERATION.

$$x + y = 5 \quad (1)$$
$$x^3 + y^3 = 35 \quad (2)$$

Cubing (1), $x^3 + 3x^2y + 3xy^2 + y^3 = 125$ (3)

Taking (2) from (3), $3x^2y + 3xy^2 = 90$ (4)

Factoring (4), $3xy(x + y) = 90$ (5)

Dividing (5) by (1), $3xy = 18$ (6)

Or, $xy = 6$ (7)

Combining (1) and (7), as in 2d example, $x = 3 \text{ or } 2$ (8)

$y = 2 \text{ or } 3$ (9)

5. Given $\left\{\begin{array}{l} x - y = 8 \\ x^3 - y^3 = 728 \end{array}\right\}$, to find x and y.

OPERATION.

$$x - y = 8 \quad (1)$$
$$x^3 - y^3 = 728 \quad (2)$$

Dividing (2) by (1), $x^2 + xy + y^2 = 91$ (3)

Squaring (1), $x^2 - 2xy + y^2 = 64$ (4)

Taking (4) from (3), $3xy = 27$ (5)

Or, $xy = 9$ (6)

Adding (6) to (3), $x^2 + 2xy + y^2 = 100$ (7)

Extracting square root of (7), $x + y = \pm 10$ (8)

But in (1), $x - y = 8$ (9)

Whence, $x = 9 \text{ or } -1$ (10)

And $y = 1 \text{ or } -9$ (11)

3d. HOMOGENEOUS EQUATIONS.

6. Given $\left\{\begin{array}{l} 2x^2 - xy = 6 \\ 2y^2 + 3xy = 8 \end{array}\right\}$ to find x and y.

OPERATION.

$$2x^2 - xy = 6 \qquad (1)$$

$$2y^2 + 3xy = 8 \qquad (2)$$

Assume $$x = vy \qquad (3)$$

Substituting vy in (1) $$2v^2y^2 - vy^2 = 6 \qquad (4)$$

And in (2) $$2y^2 + 3vy^2 = 8 \qquad (5)$$

From (4) $$y^2 = \frac{6}{2v^2 - v} \qquad (6)$$

From (5) $$y^2 = \frac{8}{2 + 3v} \qquad (7)$$

Equating (6) and (7) $$\frac{8}{2 + 3v} = \frac{6}{2v^2 - v} \qquad (8)$$

Reducing (9) $$8v^2 - 13v = 6 \qquad (9)$$

Whence, $$v = 2 \qquad (10)$$

Substituting v in (7) $$y^2 = 1 \qquad (11)$$

Whence, $$y = \pm 1 \qquad (12)$$

From (3) and (12) $$x = \pm 2 \qquad (13)$$

NOTE.—For simplicity, only one value of v was taken in equation (10).

232. Referring to the three classes, we find that,

1st. *When the equations are simple and quadratic, they may be solved by ordinary elimination.*

2d. *When the equations are similar, they may be solved by taking advantage of multiple forms, and of the relations existing between the sum, difference, and product of the unknown quantities.*

3d. *When the equations are homogeneous, they may be solved by the use of an auxiliary quantity.*

EXAMPLES FOR PRACTICE.

Find the values of x and y in the following equations.

1. $\left\{\begin{array}{l} 2x - y = 12 \\ x^2 + 2y = 53 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 7 \text{ or } -11, \\ y = 2 \text{ or } -34. \end{array}\right.$

2. $\left\{\begin{array}{l} x - 3y = 1 \\ x^2 - 3y^2 = 13 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 4 \text{ or } -5, \\ y = 1 \text{ or } -2. \end{array}\right.$

3. $\left\{\begin{array}{l} x + y = 12 \\ xy = 35 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 5 \text{ or } 7, \\ y = 7 \text{ or } 5. \end{array}\right.$

4. $\left\{\begin{array}{l} x - y = -1 \\ xy = 42 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 6 \text{ or } -7, \\ y = 7 \text{ or } -6. \end{array}\right.$

5. $\left\{\begin{array}{l} x + y = 1125 \\ x^2 - y^2 = 1125 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 563, \\ y = 562. \end{array}\right.$

6. $\left\{\begin{array}{l} x - y = 4 \\ x^3 - y^3 = 124 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 5 \text{ or } -1, \\ y = 1 \text{ or } -5. \end{array}\right.$

7. $\left\{\begin{array}{l} x^3 + y^3 = 19(x + y) \\ x - y = 3 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 5 \text{ or } -2, \\ y = 2 \text{ or } -5. \end{array}\right.$

8. $\left\{\begin{array}{l} \dfrac{1}{x} + \dfrac{1}{y} = \dfrac{5}{6} \\ \dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{13}{36} \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 2 \text{ or } 3, \\ y = 3 \text{ or } 2. \end{array}\right.$

NOTE.—Put $\frac{1}{x} = P$, and $\frac{1}{y} = Q$.

9. $\left\{\begin{array}{l} x^3 + y^3 = 152 \\ x + y = 8 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 5 \text{ or } 3, \\ y = 3 \text{ or } 5. \end{array}\right.$

10. $\left\{\begin{array}{l} x^2 + y^2 - x - y = 78 \\ xy + x + y = 39 \end{array}\right\}.$ *Ans.* $\left\{\begin{array}{l} x = 9 \text{ or } 3, \\ y = 3 \text{ or } 9. \end{array}\right.$

11. $\left\{\begin{array}{l} 2x^2 - 3xy = 50 \\ x^2 - 2y^2 = 50 \end{array}\right\}.$

Ans. $\left\{\begin{array}{l} x = 10 \text{ or } -5\sqrt{-2}, \\ y = 5 \text{ or } \ \ 5\sqrt{-2}. \end{array}\right.$

PROBLEMS

PRODUCING QUADRATICS WITH TWO UNKNOWN QUANTITIES.

NOTE.—In several of the examples which follow, the pupil may have the choice of using two symbols, or one, in the solution. It will be useful to solve by both methods.

233. 1. Divide 100 into two such parts, that the sum of their square roots may be 14. *Ans.* 64 and 36.

2. Divide the number 14 into two such parts, that the sum of the squares of those parts shall be 100. *Ans.* 8 and 6.

3. Divide the number a into two such parts, that the sum of the squares of those parts shall be b.

Ans. $\frac{1}{2}(a \pm \sqrt{2b - a^2})$.

4. It is required to divide the number 24 into two such parts, that their product may be equal to 35 times their difference. *Ans.* 10 and 14.

5. The sum of two numbers is 8, and the sum of their cubes 152; what are the numbers? *Ans.* 3 and 5.

6. Find two numbers, such that the less may be to the greater as the greater is to 12, and that the sum of their squares may be 45. *Ans.* 3 and 6.

7. What two numbers are those, whose difference is 3, and the difference of their cubes 189? *Ans.* 3 and 6.

8. What two numbers are those, whose sum is 5, and the sum of their cubes 35? *Ans.* 2 and 3.

9. A merchant has a piece of broadcloth and a piece of silk. The number of yards in both is 110; and if the square of the number of yards of silk be subtracted from 80 times the number of yards of broadcloth, the difference will be 400. How many yards are there in each piece?

Ans. 60 of silk; 50 of broadcloth.

10. A is 4 years older than B; and the sum of the squares of their ages is 976. What are their ages?

Ans. A's age, 24 years; B's, 20 years

11. Divide the number 10 into two such parts, that the square of 4 times the less part may be 112 more than the square of 2 times the greater. *Ans.* 4 and 6.

12. Find two numbers, such that the sum of their squares may be 89, and their sum multiplied by the greater may produce 104. *Ans.* 5 and 8.

13. What number is that which, being divided by the sum of its two digits, the quotient $6\frac{2}{5}$; but when 9 is subtracted from it, there remains a number having the same digits inverted? *Ans.* 32.

14. Divide 20 into three parts, such that the continued product of all three may be 270, and that the difference of the first and second may be 2 less than the difference of the second and third. *Ans.* 5, 6, and 9.

15. A regiment of soldiers, consisting of 1066 men, forms into two squares, one of which has four men more in a side than the other. What number of men are in a side of each of the squares? *Ans.* 21 and 25.

16. A farmer received 24 dollars for a certain quantity of wheat, and an equal sum for a quantity of barley, but at a price 25 cents less by the bushel. The quantity of barley exceeded the wheat by 16 bushels. How many bushels were there of each?

Ans. 32 bushels of wheat, and 48 of barley.

17. A laborer dug two trenches, one of which was 6 yards longer than the other, for 17 pounds 16 shillings, and the digging of each trench cost as many shillings per yard as there were yards in its length. What was the length of each?

Ans. 10 and 16 yards.

18. A and B set out from two towns distant from each other 247 miles, and traveled the direct road till they met. A went 9 miles a day, and the number of days at the end of which they met, was greater, by 3, than the number of miles which B went in a day. How many miles did each travel?

Ans. A, 117 miles; and B, 130.

19. The fore wheels of a carriage make 6 revolutions more than the hind wheels, in going 120 yards; but if the circumference of each wheel be increased 1 yard, the fore wheels will make only 4 revolutions more than the hind wheels, in the same distance: required the circumference of each wheel.

Ans. 4 and 5 yards.

20. There are two numbers whose product is 120. If 2 be added to the less, and 3 subtracted from the greater, the product of the sum and remainder will also be 120. What are the numbers? *Ans.* 15 and 8.

21. There are two numbers, the sum of whose squares exceeds twice their product, by 4, and the difference of their squares exceeds half their product, by 4; required the numbers. *Ans.* 6 and 8.

22. What two numbers are those, which being both multiplied by 27, the first product is a square, and the second the root of that square; but being both multiplied by 3, the first product is a cube, and the second the root of that cube?

Ans. 243 and 3.

23. A man bought a horse, which he sold, after some time, for 24 dollars. At this sale he lost as much per cent. upon the price of his purchase as the horse cost him. What did he pay for the horse?

Ans. He paid \$60 or \$40; the problem does not decide which sum.

24. What two numbers are those whose product is equal to the difference of their squares; and the greater number is to the less as 3 to 2? *Ans.* No such numbers exist.

25. What two numbers are those, the double of whose product is less than the sum of their squares by 9, and half of their product is less than the difference of their squares by 9?

Ans. The numbers are 9 and 12.

SECTION V.

ARITHMETICAL PROGRESSION.

234. An **Arithmetical Progression** is a series of numbers or quantities, increasing or decreasing by the same difference, from term to term. Thus, 2, 4, 6, 8, 10, 12, &c., is an increasing or ascending arithmetical series, having a common difference of 2; and 20, 17, 14, 11, 8, &c., is a decreasing series, whose common difference is 3.

235. The **Extremes** are the first and last terms of the series.

236. The **Means** are the intermediate terms.

CASE I.

237. To find the last term.

To investigate the properties of an arithmetical progression, let a represent the first term of a series, and d the common difference. Then

$$a, (a + d), (a + 2d), (a + 3d), (a + 4d), \&c.,$$

represents an ascending series; and

$$a, (a - d), (a - 2d), (a - 3d), (a - 4d), \&c.,$$

represents a descending series. And we observe,

1st. *The first term, a, is taken once in every term.*

2d. *The coefficient of d in any term is one less than the number of the term counted from the left.* Therefore the tenth term would be expressed by

$$a + 9d;$$

The 17th term by	$a + 16d$;
The 53d term by	$a + 52d$;
The nth term by	$a + (n - 1)d$.

When the series is descending, the sign to the term containing d will be minus, the 20th term, for example, would be

$$a - 19d$$

The nth term $\qquad a - (n - 1)d.$

If we suppose the series to terminate at the nth term, and represent this last term by L, we shall have the general formula,

$$L = a \pm (n - 1)d \qquad \text{(A)}$$

in which the *plus* sign answers to an increasing, and the *minus* sign to a decreasing series. Hence, to find the last term, we have the following

RULE. I. *Multiply the common difference by the number of terms, less one.*

II. *Add the product to the first term, when the progression is an increasing series, and subtract it from the first term when the progression is a decreasing series.*

EXAMPLES FOR PRACTICE.

If the series be increasing,

1. When $a = 2$ and $d = 3$, what is the tenth term? *Ans.* 29.
2. When $a = 3$ and $d = 2$, what is the 12th term? *Ans.* 25.
3. When $a = 7$ and $d = 10$, what is the 21st term? *Ans.* 207.
4. When $a = 1$ and $d = \frac{1}{2}$, what is the 100th term? *Ans.* $50\frac{1}{2}$.
5. When $a = 3$ and $d = \frac{1}{3}$, what is the 100th term? *Ans.* 36.
6. When $a = 0$ and $d = \frac{1}{8}$, what is the 89th term? *Ans.* 11.

If the series be decreasing,

7. When $a = 56$ and $d = 3$, what is the 15th term? *Ans.* 14.

8. When $a = 60$ and $d = 7$, what is the 9th term?

Ans. 4.

9. When $a = 325$ and $d = 16$, what is the 13th term?

Ans. 133.

10. When $a = 6$ and $d = \frac{1}{2}$, what is the 20th term?

Ans. $-3\frac{1}{2}$.

11. When $a = 30$ and $d = 3$, what is the 31st term?

Ans. -60.

CASE II.

238. To find the sum of the series.

It is manifest, that the sum of the terms will be the same, in whatever order the terms are written.

Take, for instance, the series	3,	5,	7,	9,	11.
And the same inverted,	11,	9,	7,	5,	3.
The sums of the terms will be	14,	14,	14,	14,	14.

$$\begin{array}{lccccc}
\text{Take} & a & a + d, & a + 2d, & a + 3d, & a + 4d, \quad (1)\\
\text{Inverted,} & a + 4d, & a + 3d, & a + 2d, & a + d, & a \\
\hline
\text{Sums,} & 2a + 4d, & 2a + 4d, & 2a + 4d, & 2a + 4d, & 2a + 4d. \quad (2)
\end{array}$$

This resulting series (2) is uniform; and we observe that each term is formed by adding either the extremes of the given series (1), or terms equally distant from the extremes; and that the sum of its terms must be twice that of the given series. Hence, in an arithmetical progression,

I. *The sum of the extremes is equal to the sum of any other two terms equally distant from the extremes.*

II *Twice the sum of the series is equal to the sum of the extremes taken as many times as there are terms.*

If, therefore, S represent the sum of a series, n the number of terms, a the first term, and L the last term, we shall have

$$2S = n(a + L)$$

Or,
$$S = \frac{n}{2}(a + L) \qquad \text{(B)}$$

Hence, to find the sum of the series, we have the following

RULE. *Multiply the sum of the extremes by half the number of terms.*

EXAMPLES FOR PRACTICE.

1. The first term of an arithmetical series is 5, the last term 92. and the number of terms 30; what is the sum of the terms? *Ans.* 1455.

2. The first term of an arithmetical series is 2, the number of terms 10, and the last term 30; what is the sum of the terms? *Ans.* 160.

3. The first term of an arithmetical series is 5, the number of terms 35, and the last term 107; what is the sum of the terms? *Ans.* 1960.

4. The first term of an arithmetical series is 7, the last term 207, and the number of terms 21; what is the sum of the terms? *Ans.* 2247.

5. The first term of an arithmetical series is 6, the last term $-3\frac{1}{2}$, and the number of terms 20; what is the sum of the terms? *Ans.* 25.

GENERAL APPLICATIONS.

239. In an arithmetical progression there are five parts, as follows:

1st. The first term;	Symbol	a.
2d. The common difference;	"	d.
3d. The number of terms;	"	n.
4th. The last term;	"	L.
5th. The sum of the terms;	"	S.

The formulas,

$$L = a \pm (n - 1)d, \quad \text{(A)}$$

$$S = \frac{n}{2}(a + L), \quad \text{(B)}$$

contain all of the parts enumerated above; and since these *equations* are independent of each other, they are sufficient to

determine any *two* of the parts which are *unknown*, provided the other *three* are known.

1. The sum of an arithmetical series is 1455, the first term 5, and the number of terms 30; what is the common difference?

OPERATION.

$$S = 1455$$
$$a = 5$$
$$n = 30$$
$$L = 5 + 29d \quad (1)$$
$$1455 = 15(5 + L) \quad (2)$$
$$15L = 1380 \quad (3)$$
$$L = 92 \quad (4)$$
$$5 + 29d = 92 \quad (5)$$
$$29d = 87 \quad (6)$$
$$d = 3 \quad (7)$$

ANALYSIS. The example gives 1455, 5 and 3, for the values of S, a and n respectively. We substitute the given values of a and n in formula (A) and obtain (1); likewise, substituting the values of S and n in (B), gives (2) We thus obtain two equations with two unknown quantities, d and L. Reducing (2), gives (3) from which we obtain L=92. Equating the two values of L in (1) and (4) we have (5), which reduced, gives $d = 3$.

NOTE. — If only the last term had been required, formula (B) would have been sufficient.

2. The sum of an arithmetical series is 567, the first term 7, and the common difference 2; what is the number of terms?

OPERATION.

$$S = 567$$
$$a = 7$$
$$d = 2$$
$$L = 7 + 2n - 2 \quad (1)$$
$$567 = \frac{n}{2}(7 + L) \quad (2)$$
$$L = 5 + 2n \quad (3)$$
$$567 = \frac{n}{2}(12 + 2n) \quad (4)$$
$$n^2 + 6n = 567 \quad (5)$$
$$n^2 + (\ ) + 9 = 576 \quad (6)$$
$$n + 3 = 24 \quad (7)$$
$$n = 21 \quad (8)$$

ANALYSIS. Substituting the values of a and d in formula (A), gives (1), and the values of S and a in formula (B), gives (2). Reducing (1) we obtain (3). Substituting this value of L in (2), we obtain (4), which cleared of fractions, gives (5). Completing the square and reducing, we have, finally, $n = 21$, the number of terms.

Hence, for the determination of any required parts in an arithmetical progression, we have the following

RULE. *Substitute, in the formulas* (A) *and* (B), *the known quantities given in the example, and reduce the resulting equations.*

PROBLEMS.

3. Find *seven* arithmetical means between 1 and 49.

NOTE.—If there are 7 *means*, there must be 9 *terms*; hence,

$$n = 9,\ a = 1, \text{ and}$$
$$L = 49.$$

Ans. 7, 13, 19, 25, 31, 37, 43.

4. The first term of an arithmetical series is 1, the sum of the terms 280, and the number of terms 32; what is the common difference, and what the last term?

Ans. $d = \frac{1}{2}$, $L = 16\frac{1}{2}$.

5. Insert three arithmetical means between $\frac{1}{3}$ and $\frac{1}{2}$.

Ans. The means are $\frac{3}{8}$, $\frac{5}{12}$, $\frac{11}{24}$.

6. Insert five arithmetical means between 5 and 15.

Ans. The means are $6\frac{2}{3}$, $8\frac{1}{3}$, 10, $11\frac{2}{3}$, $13\frac{1}{3}$.

7. Suppose 100 balls be placed in a straight line, at the distance of a yard from each other; how far must a person travel to bring them one by one to a box placed at the distance of a yard from the first ball?

Ans. 5 miles 1300 yards.

8. A speculator bought 47 building lots in a certain village, giving 10 dollars for the first, 30 dollars for the second, 50 dollars for the third, and so on; what did he pay for the whole 47? *Ans.* $22090.

9. In gathering up a certain number of balls, placed on the ground in a straight line, at the distance of 2 yards from each other, the first being placed 2 yards from the box in which they were deposited, a man, starting from the box, traveled 11 miles and 840 yards; how many balls were there?

Ans. 100.

10. How many strokes do the clocks of Venice, which go on to 24 o'clock, strike in a day? *Ans.* 300.

11. In a descending arithmetical series, the first term is 730, the common difference 2, and the last term 2; what is the number of terms? *Ans.* 365.

12 The sum of the terms of an arithmetical series is 280, the first term 1, and the number of terms 32; what is the common difference? *Ans.* $\frac{1}{2}$.

13. The sum of the terms of an arithmetical series is 950, the common difference 3, and the number of terms 25; what is the first term? *Ans.* 2.

14. What is the sum of n terms of the series 1, 2, 3, 4, 5, &c.?

$$Ans.\ S = \frac{n}{2}(1 + n).$$

PROBLEMS IN ARITHMETICAL PROGRESSION

TO WHICH THE PRECEDING FORMULAS, (A) AND (B), DO NOT IMMEDIATELY APPLY.

240. Since the sum of the extremes, in an arithmetical series, is equal to the sum of any two terms equally distant from the extremes (**238**, I), we have the following special properties:

I. *When three quantities are in arithmetical progression, the mean is equal to half the sum of the extremes.*

II. *When four quantities are in arithmetical progression, the sum of the means is equal to the sum of the extremes.*

Take, for example, any three consecutive terms of a series, as

$$a + 2d,\ a + 3d,\ a + 4d,$$

and we perceive, by inspection, that the sum of the extremes is double the mean, or the mean is half the sum of the extremes.

Take four consecutive terms,

and we have $a + 2d,\ a + 3d,\ a + 4d,\ a + 5d,$

$(a + 2d) + (a + 5d) = 2a + 7d$, sum of the extremes;

$(a + 3d) + (a + 4d) = 2a + 7d$, sum of the means

To facilitate the solution of problems, when three terms are in question, let them be represented by $(x - y)$, x, $(x + y)$, y being the common difference.

When four numbers are in question, let them be represented by $(x - 3y)$, $(x - y)$, $(x + y)$, $(x + 3y)$, $2y$ being the common difference.

Such notation will often secure the formation of *similar equations;* and in applying the principles enunciated on the preceding page, the *common difference will disappear by addition.*

1. Three numbers are in arithmetical progression; the product of the first and second is 15, and of the first and third is 21; what are the numbers?

SOLUTION.

Let $y =$ common difference,
$x - y =$ first term,
$x =$ second term,
$x + y =$ third term.

$$x(x - y) = 15 \qquad (1)$$
$$x^2 - y^2 = 21 \qquad (2)$$
$$\frac{x + y}{x} = \frac{7}{5} \qquad (3)$$
$$x = \frac{5y}{2} \qquad (4)$$
$$\frac{25y^2}{4} - y^2 = 21 \qquad (5)$$
$$21y^2 = 21 \times 4 \qquad (6)$$
$$y^2 = 4 \qquad (7)$$
$$y = 2 \qquad (8)$$
$$x = 5, \text{ 2d term}, \qquad (9)$$
$$x - y = 2, \text{ 1st term}, \qquad (10)$$
$$x + y = 7, \text{ 3d term}, \qquad (11)$$

ANALYSIS. We represent the common difference by y, and the first term by $x - y$. Then x must be the second term, and $x + y$, the third term. From the given conditions we have (1) and (2). Dividing (2) by (1) we obtain (3), which reduced to find the value of x, gives (4). Substituting this value of x in (2), we have (5), and reducing this equation, we find $y = 2$. Hence, from (4), we get $x = 5$, &c.

2. There are four numbers, in arithmetical progression; the sum of the two means is 25, and the second number, multiplied by the common difference is 50; what are the numbers?

Ans. 5, 10, 15, and 20.

3. There are four numbers, in arithmetical progression; the product of the first and third is 5, and of the second and fourth is 21; what are the numbers?

Ans. 1, 3, 5, and 7.

4. There are five numbers, in arithmetical progression; the sum of these numbers is 65, and the sum of their squares 1005; what are the numbers?

NOTE.—Let $x =$ the middle term, and y the common difference. Then $x - 2y$, $x - y$, x, $x + y$, $x + 2y$, will represent the numbers.

Ans. 5, 9, 13, 17, and 21.

5. The sum of three numbers in arithmetical progression is 15, and their continued product is 105; what are the numbers? *Ans.* 3, 5, and 7.

6. There are three numbers, in arithmetical progression; their sum is 18, and the sum of their squares 158; what are the numbers? *Ans.* 1, 6, and 11.

7. Find three numbers, in arithmetical progression, such that the sum of their squares shall be 56, and the sum arising from adding together once the first and twice the second, and thrice the third, shall amount to 28. *Ans.* 2, 4, 6.

8. Find three numbers, such that their sum may be 12, and the sum of their fourth powers 962; and the numbers have equal differences in order from the least to the greatest.

Ans. 3, 4, 5.

9. Find three numbers having equal differences, and such that the square of the least number added to the product of the two greater, may make 28, but the square of the greatest number added to the product of the two less, may make 44.

Ans. 2, 4, 6.

10. Find three numbers, in arithmetical progression, such that their sum shall be 15, and the sum of their squares 93.

Ans. 2, 5, and 8.

11. Find three numbers, in arithmetical progression, such that the sum of the first and third shall be 8, and the sum of the squares of the second and third shall be 52.

Ans. 2, 4, and 6.

12. Find four numbers, in arithmetical progression, such that the sum of the first and fourth shall be 13, and the difference of the squares of the two means shall be 39.

Ans. 2, 5, 8, and 11.

13. Find seven numbers, in arithmetical progression, such that the sum of the first and sixth shall be 14, and the product of the third and fifth shall be 60.

Ans. 2, 4, 6, 8, 10, 12, and 14.

14. Find five numbers, in arithmetical progression, such that their sum shall be 25, and their continued product 945.

Ans. 1, 3, 5, 7, and 9.

15. Find four numbers, in arithmetical progression, such that the difference of the squares of the first and second shall be 12, and the difference of the squares of the third and fourth shall be 28. *Ans.* 2, 4, 6, and 8.

GEOMETRICAL PROGRESSION.

241. A **Geometrical Progression** is a series of numbers increasing or decreasing by a constant multiplier. Thus 2, 6, 18, 54, 162, &c., is a geometrical series, in which the first term is 2, and the multiplier is 3.

The series 9, 3, 1, $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, &c., is also a *geometrical series*, in which the first term is 9, and the *multiplier* is $\frac{1}{3}$.

When the multiplier is greater than 1, the series is *ascending*, and when the multiplier is less than 1, the series is *descending*.

242. The **Ratio** is the constant multiplier.

NOTE. — The words, *means* and *extremes*, as defined in Arithmetical Progression, have the same application in Geometrical progression, or in any series.

CASE I.

243. To find the last term.

If we represent the first term of the series by a, and the ratio by r, then

$$a, ar, ar^2, ar^3, ar^4, \text{\&c.},$$

will represent the series; and we observe,

1st. *The first term is a factor in every term.*

2d. *The exponent of r in any term is one less than the number of the term*; thus, in the second term it is 1; in the third term, 2; in the fourth term, 3; and in the nth term it must be $n - 1$.

Therefore, if n represent the number of terms in any series, and L the last term, then

$$L = ar^{n-1} \qquad (A).$$

23

Hence, the following

RULE. I. *Raise the ratio to a power whose index is one less than the number of terms.*

II. *Multiply the result by the first term.*

NOTE.—When any two successive terms in the series are given, it is evident that the ratio may be found by dividing any term by the preceding term.

EXAMPLES FOR PRACTICE.

1. The first term of a geometrical series is 3, and the ratio is 2; what is the 6th term of the series?

Ans. $L = (2)^5 \times 3 = 96$.

2. The first term of a series is 5, and the ratio 4; what is the 9th term?

Ans. 327680.

3. The first term of a series is 2, and the ratio 3; what is the 8th term?

Ans. 4374.

4. The first term of a series is 1, and the ratio $\frac{3}{4}$; what is the 5th term?

Ans. $\frac{81}{256}$.

5. The first term of a series is 2, and the ratio 3; what is the 7th term?

Ans. 1458.

6. The first term of a geometrical series is 5, and the ratio 4; what is the 6th term?

Ans. 5120.

7. The first term of a series is 1 and the ratio 2; what is the 10th term?

Ans. 512.

8. The first term of a series is 1, and the ratio $\frac{2}{3}$; what is the 8th term?

Ans. $\frac{128}{2187}$.

CASE II.

244. To find the sum of the series.

Let S represent the sum of any geometrical series; then we have

$$S = a + ar + ar^2 + ar^3, \text{ \&c., to } ar^{n-1}. \quad (1)$$

Multiply this equation by r, and we have

$$rS = ar + ar^2 + ar^3, \text{ \&c., to } ar^{n-1} + ar^n. \quad (2)$$

Subtracting (1) from (2), we have

$$(r - 1)S = ar^n - a. \quad (3)$$

But from Case I. we have

$$L = ar^{n-1},$$

And multiplying by r,

$$rL = ar^n.$$

Therefore, by substituting the values of ar^n in (3),

$$(r - 1)S = rL - a.$$

Or,

$$S = \frac{rL - a}{r - 1}. \quad (B)$$

Hence, the following

RULE. *Multiply the last term by the ratio, and from the product subtract the first term, and divide the remainder by the ratio less one.*

EXAMPLES FOR PRACTICE.

1. The first term is 5, the last term 1280, and the ratio 4; what is the sum of the series?

$$Ans.\ S = \frac{1280 \times 4 - 5}{3} = 1705.$$

2. The first term is 2, the last term 486, and the ratio 3; what is the sum of the series? Ans. 728.

3. The first term is 12, the last term 7500, and the ratio 5; what is the sum of the series? *Ans.* 9372.

NOTE.—If the last term is not given, it may first be found by Case I.

4. What is the sum of 8 terms of the series, 2, 6, 18, &c.
Ans. 6560.

5. What is the sum of 10 terms of the series, 4, 12, 36, &c.?
Ans. 118096.

6. What is the sum of 9 terms of the series, 5, 20, 80, &c.?
Ans. 436905.

7. What is the sum of 5 terms of the series, 3, $4\frac{1}{2}$, $6\frac{3}{4}$, &c.?
Ans. $39\frac{9}{16}$.

8. What is the sum of 10 terms of the series, 1, $\frac{2}{3}$, $\frac{4}{9}$, &c.?
Ans. $\frac{58025}{19683}$.

9. A man purchased a house, giving 1 dollar for the first door, 2 dollars for the second, 4 dollars for the third, and so on; what did the house cost him, there being 10 doors?
Ans. $1023.

10. A farmer planted 1 bushel of corn, and it produced 20 bushels. The next year he planted the 20 bushels, and they produced at the same rate as the first bushel. If he planted each year's crop successively for 5 years, and it produced at the same rate, what would be the amount of the 5 years' harvest?

NOTE.—As the 1 bushel was planted and the 20 bushels were harvested the first year, the number of terms is 6.

Ans. 3368420 bushels.

INFINITE SERIES.

245. By formula (B), and the rule subsequently given, we perceive that the sum of the series depends on the first and last *terms* and the ratio. Suppose, now, that we are required to find

the sum of a descending series, as 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, &c. We perceive that the terms decrease in value as the series advances; the hundreth term would be extremely small, the thousandth term would be very much less, and the *infinite term* nothing.

Therefore, as L becomes 0, it follows that in any decreasing series, when the number of terms is *conceived* to be infinite, the sum of the series depends wholly on the *first term* and the *ratio;* and equation (B) becomes

$$S = \frac{-a}{r-1}$$

By change of signs $\quad S = \dfrac{a}{1-r}$.

This gives for the sum of a decreasing infinite series, the following

RULE. *Divide the first term by the difference between unity and the ratio.*

EXAMPLES FOR PRACTICE.

1. Find the value of 1, $\frac{3}{4}$, $\frac{9}{16}$, &c., to infinity.

NOTE. $a = 1$, $r = \frac{3}{4}$.

Ans. 4.

2. Find the exact value of the series, 2, 1, $\frac{1}{2}$, &c., to infinity.
Ans. 4.

3. Find the exact value of the series, 6, 4, &c., to infinity.
Ans. 18.

4. Find the exact value of the decimal .3333, &c., to infinity.

NOTE.—This circulating decimal may be expressed thus: $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} +$, &c. Hence, $a = \frac{3}{10}$, and $r = \frac{1}{10}$. In all cases of this kind, the *repetend*, taken with its local value, will be the first term; and the order of its lowest figure will indicate the fractional ratio.

Ans. $\frac{1}{3}$.

5. **Find the value of .323232, &c., to infinity.**

$a = \frac{32}{100}$, $ar = \frac{32}{10000}$; therefore, $r = \frac{1}{100}$. *Ans.* $\frac{32}{99}$.

6. **Find the value of .777, &c., to infinity.** *Ans.* $\frac{7}{9}$.

7. **Find the sum of the infinite series,** $1 + \frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} +$, &c. *Ans.* $\frac{x^2}{x^2 - 1}$.

8. **Find the sum of the infinite geometrical progression,** $a - b + \frac{b^2}{a} - \frac{b^3}{a^2} + \frac{b^4}{a^3} -$, &c., in which the ratio is $-\frac{b}{a}$.

Ans. $\frac{a^2}{a + b}$.

GEOMETRICAL MEANS.

246. If we take any three terms in geometrical progression, as

$$a, ar, ar^2,$$

we have

$$a \times ar^2 = a^2r^2, \text{ product of the extremes;}$$
$$(ar)^2 = a^2r^2, \text{ square of the mean.}$$

If we take four terms,

$$a, ar, ar^2, ar^3,$$

we have

$$a \times ar^3 = a^2r^3, \text{ product of the extremes;}$$
$$ar \times ar^2 = a^2r^3, \text{ product of the means.}$$

Hence,

I. *When three numbers are in geometrical progression, the product of the extremes is equal to the square of the mean.*

II. *When four numbers are in geometrical progression, the product of the extremes is equal to the product of the means.*

NOTE.—This last property belongs also to geometrical *proportion* But the pupil must not confound *proportion* with series.

$a : ar :: b : br$, are quantities in *geometrical proportion.*
$a : ar : ar^2 : ar^3$, are quantities in *geometrical progression.*

247. To find the geometrical mean of two numbers, we have, from the principle enunciated above (I), the following

RULE. *Multiply the extremes together, and take the square root of the product.*

EXAMPLES FOR PRACTICE.

1. Find the geometrical mean between 2 and 8.

Ans. $\sqrt{2 \times 8} = 4.$

2. Find the geometrical mean between 3 and 12.

Ans. 6.

3. Find the geometrical mean between 5 and 80.

Ans. 20.

4. Find the geometrical mean between a and b.

Ans. $(ab)^{\frac{1}{2}}$.

5. Find the geometrical mean between $\frac{1}{4}$ and 9. *Ans.* $\frac{3}{2}$.

6. Find the geometrical mean between $3a$ and $27a$.

Ans. $9a$.

7. Find the geometrical mean between 1 and 9. *Ans.* 3.

8. Find the geometrical mean between 2 and 3.

Ans. $\sqrt{6}$.

9. Find the geometrical mean between $\sqrt[3]{a^2x}$ and $\sqrt[3]{ax^2}$.

Ans. $\sqrt{ax}$.

APPLICATIONS.

248. In a geometrical progression there are five parts, as follows:

1st.	The first term;	symbol	a.
2d.	The ratio;	"	r.
3d.	The number of terms;	"	n.
4th.	The last term;	"	L.
5th.	The sum of the terms;	"	S.

Since the independent equations,

$$L = ar^{n-1} \qquad \text{(A)}$$

$$S = \frac{rL - a}{r - 1} \qquad \text{(B)}$$

contain all of the parts, any *two* may be determined when the other *three* are known.

NOTE.—Since n enters into the above formulas only as an *exponent*, the process of determining it requires a knowledge of logarithms, and must be omitted here.

1. The sum of a geometrical progression is 468, the number of terms is 4, and the ratio 5; what is the first term?

OPERATION.

$$S = 468$$
$$n = 4$$
$$r = 5$$

$$L = 125a \qquad (1)$$

$$468 = \frac{5L - a}{4} \qquad (2)$$

$$468 = \frac{625a - a}{4} \qquad (3)$$

$$156a = 468 \qquad (4)$$

$$a = 3 \qquad (5)$$

ANALYSIS. From the example, we have 468, 4, and 5, for the values of S, n, and r, respectively; $n - 1$ is 3, and r^{n-1} is, therefore, equal to 125, which substituted in formula (A), gives (1). The value of r, substituted in formula (B), gives (2). To eliminate L we substitute its value, $125a$, in (2), and obtain (3), which reduced, gives $a = 3$.

2. The sum of a geometrical series is 847, the ratio 3, and the number of terms 5; what is the first term? *Ans.* 7.

3. The sum of a geometrical progression is 6220, the ratio 6, and the number of terms 5; what is the last term?

Ans. 5184.

4. The first and last terms of a geometrical series are 2 and 162, and the number of terms 5; required the ratio.

NOTE.—From formula (A) we have $r = \left(\frac{L}{a}\right)^{\frac{1}{n-1}}$.

Ans. 3.

5. The first term of a geometrical series is 28, the last term 17500, and the number of terms 5; what is the ratio?

Ans. 5.

6. The first term of a geometrical series is 32, the last term 4000, and the number of terms 4; what is the ratio?

Ans. 5.

7. Find two geometrical means between 4 and 256.

NOTE.—Two means will require *four* terms. Hence $n = 4$, from which data r may be found, as above.

Ans. 16 and 64.

8. Find three geometrical means between 1 and 16.

Ans. 2, 4, and 8.

PROBLEMS IN GEOMETRICAL PROGRESSION

TO WHICH THE FORMULAS (A) AND (B) DO NOT DIRECTLY APPLY.

249. The general representation of the terms of a geometrical progression is made thus:

$$x,\ xy,\ xy^2,\ xy^3,\ \&c.,$$

in which x is the first term, and y the ratio. But when particular relations are given, it may become necessary to adopt a different *notation.*

When *three* terms are considered in the problem, they may be represented thus:

$$x,\ \sqrt{xy},\ y;$$
$$\text{Or,}\quad x^2,\ xy,\ y^2.$$

Since, in each case, the product of the extremes is equal to the square of the mean (**246**, I).

When there are *four* terms, we may adopt the following notation:

$$\frac{x^2}{y},\ x,\ y,\ \frac{y^2}{x};$$

for the product of the extremes is equal to the product of the means (**246**, II).

1. The sum of three numbers in geometrical progression is 7, and the sum of their squares is 21; what are the numbers?

SOLUTION.

First condition,	$x + \sqrt{xy} + y = 7$	(1)
Second condition,	$x^2 + xy + y^2 = 21$	(2)
From (1),	$x + y = 7 - \sqrt{xy}$	(3)
From (2),	$x^2 + y^2 = 21 - xy$	(4)
Squaring (3),	$x^2 + 2xy + y^2 = 49 - 14\sqrt{xy} + xy$	(5)
Taking (4) from (5),	$2xy = 28 - 14\sqrt{xy} + 2xy$	(6)
Reducing (6),	$\sqrt{xy} = 2$, the mean,	(7)
From (7),	$3xy = 12$	(8)
Taking (8) from (2),	$y^2 - 2xy + x^2 = 9$	(9)
Extracting square root of (9),	$y - x = 3$	(10)
From (3) and (7),	$y + x = 5$	(11)
From (10) and (11)	$x = 1$, first term,	(12)
And,	$y = 4$, third term,	(13)

NOTE.—In equation (9), y^2 is put first because y is greater than x.

2. The product of three numbers in geometrical progression is 64, and the sum of their cubes is 584; what are the numbers?

SOLUTION.

Let	$x^2,\ xy,\ y^2$ be the numbers,	
First condition,	$x^3y^3 = 64$	(1)
Second condition,	$x^6 + x^3y^3 + y^6 = 584$	(2)
Adding (1) to (2),	$x^6 + 2x^3y^3 + y^6 = 648\ (= 324 \cdot 2)$	(3)
Extracting the square root,	$x^3 + y^3 = 18\sqrt{2}$	(4)
Taking 3 times (1) from (2),	$x^6 - 2x^3y^3 + y^6 = 392\ (= 196 \cdot 2)$	(5)
Square root,	$y^3 - x^3 = 14\sqrt{2}$	(6)
Equation (4),	$y^3 + x^3 = 18\sqrt{2}$	(7)
From (6) and (7),	$x^3 = 2\sqrt{2}$	(8)
Squaring (8),	$x^6 = 8$	(9)
Cube root,	$x^2 = 2$, 1st term,	(10)
From (6) and (7),	$y^3 = 16\sqrt{2}$	(11)
Squaring (11),	$y^6 = 512$	(12)
Cube root,	$y^2 = 8$, 3d term,	(13)
From (10) and (13),	$x^2y^2 = 16$	(14)
Square root,	$xy = 4$, 2d term.	(15)

3. Of three numbers in geometrical progression, the sum of the first and second is 90, and the sum of the second and third is 180; what are the numbers?

NOTE. Represent the numbers by x, xy, and xy^2.

Ans. 30, 60, and 120.

4. The sum of the first and third of four numbers in geometrical progression is 20, and the sum of the second and fourth is 60; what are the numbers? *Ans.* 2, 6, 18, 54.

5. Divide the number 210 into three parts, such that the last shall exceed the first by 90, and the parts be in geometrical *progression.* *Ans.* 30, 60, and 120.

6. The sum of four numbers in geometrical progression is 80, and the last term divided by the sum of the mean terms is $1\frac{1}{3}$; what are the numbers? *Ans.* 2, 4, 8, and 16.

7. The sum of the first and third of four numbers in geometrical progression is 148, and the sum of the second and fourth is 888; what are the numbers?

Ans. 4, 24, 144, and 864

8. The continued product of three numbers in geometrical progression is 216, and the sum of the squares of the extremes is 328; what are the numbers? *Ans.* 2, 6, 18.

9. The sum of three numbers in geometrical progression is 13, and the sum of the extremes being multiplied by the mean, the product is 30; what are the numbers?

Ans. 1, 3, and 9.

10. Of three numbers in geometrical progression, the sum of the first and last is 52, and the square of the mean is 100; what are the numbers? *Ans.* 2, 10, 50.

11. There are three numbers in geometrical progression; their sum is 31, and the sum of the squares of the first and last is 626; what are the numbers? *Ans.* 1, 5, 25.

12. It is required to find three numbers in geometrical progression, such that their sum shall be 14, and the sum of their squares 84. *Ans.* 2, 4, and 8.

13. Of four numbers in geometrical progression, the second number is less than the fourth by 24, and the sum of the extremes is to the sum of the means, as 7 to 3; what are the numbers? *Ans.* 1, 3, 9, and 27.

14. The sum of four numbers in geometrical progression is equal to their common ratio + 1, and the first term is $\frac{1}{10}$; what are the numbers? *Ans.* $\frac{1}{10}$, $\frac{3}{10}$, $\frac{9}{10}$, $\frac{27}{10}$.

15. How much will $500 amount to in 4 years, at 7 per cent. compound interest?

NOTE. — Since we have the principal at the commencement, and the first year's amount at the end of the first year, the number of terms is 5. The ratio is $1.07, equal to the amount of $1.00 for one year.

Ans. $655.398.

PROPORTION.

250. Two quantities of the same kind may be compared, and their numerical relation determined, by seeking how many times one contains the other. The relation of *four* quantities may be determined by comparing the relation of *two* of like kind, with *two others* of like kind. These comparisons give rise to *ratio* and *proportion.*

251. Ratio is the quotient of one quantity divided by another of the same kind, regarded as the standard of comparison.

There are two methods of indicating ratio.

1*st.* By writing the divisor and dividend with two dots between them; thus,

$$a : b$$

is the indicated ratio of a to b, in which a is the divisor, and b the dividend.

2*d.* In the form of a fraction; thus, the above ratio becomes

$$\frac{b}{a}$$

252. Proportion is an equality of ratios; thus, if two quantities, a and b, have the same ratio as two other quantities, c and d, the *four* quantities, a, b, c, and d, are said to be *proportional.*

Proportion is written in two ways; thus,

$$a : b :: c : d,$$

which is read, a is to b, as c is to d; or thus,

$$a : b = c : d,$$

which is read, the ratio of a to b is equal to the ratio of c to d.

NOTE.—The second is the modern method, and more fitly expresses the *nature of proportion.*

253. A **Couplet** is the two quantities which form a ratio.

254. The **Terms** of a proportion are the four quantities which are compared.

255. The **Antecedents** in a proportion are the first terms of the two couplets; or the *first* and *third* terms of the proportion.

256. The **Consequents** in a proportion are the second terms of the two couplets; or the *second* and *fourth* terms of the proportion.

257. The **Extremes** in a proportion are the *first* and *fourth* terms.

258. The **Means** in a proportion are the *second* and *third* terms.

259. A **Mean Proportional** between two quantities is a quantity to which the first of the two given quantities has the same ratio as the quantity itself has to the second; thus, if

$$a : b = b : c$$

the quantity, b, is a mean proportional between a and c; and the three quantities are said to be in *continued proportion*

260. A **Proposition** is the statement of a truth to be demonstrated, or of an operation to be performed.

261. A **Scholium** is any remark showing the application or limitation of a preceding proposition.

262. If in the proportion

$$a : b = c : d$$

the 2d method of indicating ratio be employed, we shall have

$$\frac{b}{a} = \frac{d}{c} \quad \text{(A)}$$

which is the *fundamental equation of proportion.* And any proposition relating to proportion will be *proved,* when shown to be consistent with this equation.

GENERAL PRINCIPLES OF PROPORTION.

PROPOSITION I.

263. *In every proportion, the product of the extremes is equal to the product of the means.*

Let $a : b = c : d$, represent any proportion;

Then by formula (A), $\frac{b}{a} = \frac{d}{c}$;

Clearing of fractions, $bc = ad$.

That is, the product of b and c, the means, is equal to the product of a and d, the extremes.

SCHOLIUM.—From the last equation, we have

$$\left.\begin{array}{ll}\text{The first mean,} & b = \frac{ad}{c} \\ \text{The second mean,} & c = \frac{ad}{b}\end{array}\right\} \quad (1).$$

$$\left.\begin{array}{ll}\text{The first extreme,} & a = \frac{bc}{d} \\ \text{The second extreme,} & d = \frac{bc}{a}\end{array}\right\} \quad (2).$$

Hence,

1st. *Either mean is equal to the product of the extremes divided by the other mean.* (1).

2d. *Either extreme is equal to the product of the means divided by the other extreme.* (2).

PROPOSITION II.

264. Conversely. *If the product of two quantities be equal to the product of two others, then two of them may be taken for the means, and the other two for the extremes of a proportion.*

Let $bc = ad$ (1)

Dividing by c, $b = \frac{ad}{c}$ (2)

Dividing (2) by a, $\frac{b}{a} = \frac{d}{c}$ (3)

Hence by formula (A), $a : b = c : d$

in which the factors of the first product, cb, are the means, and the factors of the second product, ad, are the extremes.

PROPOSITION III.

265. *If four quantities be in proportion, they will be in proportion by* ALTERNATION; *that is, the antecedents will be to each other as the consequents.*

Let $a : b = c : d$

Then by formula (A), $\frac{b}{a} = \frac{d}{c}$ (1)

Multiplying (1) by c, $\frac{bc}{a} = d$ (2)

Dividing (2) by b, $\frac{c}{a} = \frac{d}{b}$ (3)

Hence, $a : c = b : d$

in which a and c, the antecedents of the given proportion, are proportional to b and d, the consequents of the given proportion.

SCHOLIUM.—A proportion and an equation may be regarded as but different forms for the same expression, and every equation may be converted into a proportion under various forms. For example,

$$
\begin{array}{ll}
\text{Let,} & xy = a(a+b) \\
\text{Then,} & x : a = a + b : y \\
\text{Or,} & xy : a = a + b : 1 \\
\text{Or,} & a : x = y : (a+b)
\end{array}
$$

PROPOSITION IV.

266. *If four quantities be in proportion, they will be in proportion by* INVERSION; *that is, the second will be to the first, as the fourth to the third.*

$$
\begin{array}{lll}
\text{Let} & a : b = c : d & \\
\text{Then by formula (A),} & \dfrac{b}{a} = \dfrac{d}{c} & (1) \\
\text{Clearing of fractions,} & bc = ad & (2) \\
\text{Hence by Prop. II,} & b : a :: d : c. &
\end{array}
$$

PROPOSITION V.

267. *If three quantities be in continued proportion, the product of the extremes is equal to the square of the mean.*

$$
\begin{array}{ll}
\text{Let,} & a : b = b : c \\
\text{By Prop. I,} & ac = bb = b^2
\end{array}
$$

SCHOLIUM.—Extracting the square root of the last equation, we have

$$b = \sqrt{ac}; \text{ hence,}$$

The mean proportional between two quantities is equal to the square root of their product.

PROPOSITION VI.

268. *Quantities which are proportional to the same quantities are proportional to each other.*

If	$a : b = P : Q$	(A)
And	$c : d = P : Q$	(B)
We are to prove that	$a : b = c : d$	
From (A)	$\frac{b}{a} = \frac{Q}{P}$	
Form (B)	$\frac{d}{c} = \frac{Q}{P}$	
Hence, (Ax. 7),	$\frac{b}{a} = \frac{d}{c}$	
Or,	$a : b = c : d$	

PROPOSITION VII.

269. *If four magnitudes be in proportion, they must be in proportion by* COMPOSITION *or* DIVISION; *that is, the first is to the sum of the first and second, as the third is to the sum of the third and fourth; or, the first is to the difference between the first and second, as the third is to the difference between the third and fourth.*

If	$a : b = c : d$	(A)
We are to prove that	$a : a \pm b = c : c \pm d$	
From (A),	$\frac{b}{a} = \frac{d}{c}$	(1)
Take	$1 = 1$	(2)
Adding (1) to (2),	$1 + \frac{b}{a} = 1 + \frac{d}{c}$	(3)
Subtracting (1) from (2),	$1 - \frac{b}{a} = 1 - \frac{d}{c}$	(4)
From (3),	$\frac{a + b}{a} = \frac{c + d}{c}$	(5)
From (4),	$\frac{a - b}{a} = \frac{c - d}{c}$	(6)
Hence, from (5),	$a : a + b = c : c + d$	
And from (6),	$a : a - b = c : c - d.$	

SCHOLIUM.—This *composition* may be carried to almost any extent, as we see by the following investigation:

Take the equation, $\frac{b}{a} = \frac{d}{c}$ (1)

Multiplying by $m = m$

And $\frac{mb}{a} = \frac{md}{c}$ (2)

Adding, $n = n$

And $n + \frac{mb}{a} = n + \frac{md}{c}$ (3)

Reducing, $\frac{na + mb}{a} = \frac{nc + md}{c}$ (4)

Hence, $a : na + mb = c : nc + md$

Or, subtracting (2) from $n = n$, and proceeding as before, we shall have

$$a : na - mb = c : nc - md$$

PROPOSITION VIII.

270. *If four quantities be in proportion, the sum of the two quantities which form the first couplet is to their difference, as the sum of the two quantities which form the second couplet is to their difference.*

If $a : b = c : d$ (A)

We are to prove that $a + b : a - b = c + d : c - d$

From (A), by Prop. VII, $a : a + b = c : c + d$ (1)

Also, $a : a - b = c : c - d$ (2)

From (1), $\frac{a + b}{a} = \frac{c + d}{c}$ (3)

From (2), $\frac{a - b}{a} = \frac{c - d}{c}$ (4)

Dividing (4), by (3), $\frac{a - b}{a + b} = \frac{c - d}{c + d}$ (5)

Whence $a + b : a - b = c + d : c - d$

PROPOSITION IX.

271. *If four quantities be in proportion, either couplet may be multiplied or divided by any number whatever, and the quantities will still be in proportion.*

Let $a : b = c : d$

Then, $\frac{b}{a} = \frac{d}{c}$ (1).

Multiplying both numerator and denominator of either of these fractions by any number, n, (**105**, III),

We have, $\frac{nb}{na} = \frac{d}{c}$ (2)

Also, $\frac{b}{a} = \frac{nd}{nc}$ (3)

Hence from (2), $na : nb = c : d$

And from (3), $a : b = nc : nd$;

in which, if n represent a whole number, the couplets are *multiplied;* and if n represent a fraction, the couplets are *divided.*

PROPOSITION X.

272. *If four quantities be in proportion, either the antecedents or the consequents may be multiplied by any number and the four quantities will still be in proportion.*

Let $a : b = c : d$

Then, $\frac{b}{a} = \frac{d}{c}$ (1)

Take $m = m$ (2)

Multiplying (1) by (2), (**105**, I), $\frac{mb}{a} = \frac{md}{c}$ (3)

Dividing (1) by (2), (**105**, II), $\frac{b}{ma} = \frac{d}{mc}$ (4)

Hence from (3), $a : mb = c : md$

And from (4), $ma : b = mc : d$

PROPOSITION XI.

273. *If four quantities be in proportion, like powers or roots of the same quantities will be in proportion.*

Let $a : b = c : d$

Then, $\frac{b}{a} = \frac{d}{c}$ (1)

nth power of (1), $\frac{b^n}{a^n} = \frac{d^n}{c^n}$ (2)

nth root of (1), $\frac{b^{\frac{1}{n}}}{a^{\frac{1}{n}}} = \frac{d^{\frac{1}{n}}}{c^{\frac{1}{n}}}$ (3)

Hence, from (2), $a^n : b^n = c^n : d^n$

And from (3), $a^{\frac{1}{n}} : b^{\frac{1}{n}} = c^{\frac{1}{n}} : d^{\frac{1}{n}}$

PROPOSITION XII.

274. *If four quantities in proportion be multiplied or divided term by term by four others also in proportion, the product or quotient will still form a proportion.*

If $a : b = c : d$ (A)

And $x : y = m : n$ (B)

We are to prove that

$ax : by = cm : dn$

And $\frac{a}{x} : \frac{b}{y} = \frac{c}{m} : \frac{d}{n}$

From (A), $ad = bc$ (1)

From (B), $xn = ym$ (2)

Multiplying (1) by (2), $(ax)(dn) = (by)(cm)$ (3)

Dividing (1) by (2), $\left(\frac{a}{x}\right)\left(\frac{d}{n}\right) = \left(\frac{b}{y}\right)\left(\frac{c}{m}\right)$ (4)

From (3) by Prop. II, $ax : by = cm : dn$

And from (4), $\frac{a}{x} : \frac{b}{y} = \frac{c}{m} : \frac{d}{n}$

PROPOSITION XIII.

275. *If any number of proportionals have the same ratio, any one of the antecedents will be to its consequent as the sum of all the antecedents is to the sum of all the consequents.*

Let	$a : b = a : b$	(A)
Also,	$a : b = c : d$	(B)
	$a : b = m : n$	(C)
	&c. = &c.	
We are to prove that	$a : b = (a + c + m) : (b + d + n)$	
From (A),	$ab = ab$	
From (B),	$ad = cb$	
From (C),	$an = mb$	
By addition,	$a(b + d + n) = b(a + c + m)$	
By Prop. II,	$a : b = (a + c + m) : (b + d + n)$	

PROBLEMS IN PROPORTION.

276. 1. Find two numbers, the greater of which is to the less as their sum to 42, and the greater to the less as their difference is to 6.

SOLUTION.

Let	$x =$ greater; $y =$ less.	
By the conditions,	$x : y = x + y : 42$	(1)
	$x : y = x - y : 6$	(2)
Prop. VI,	$x + y : 42 = x - y : 6$	(3)
Prop. III,	$x + y : x - y = 42 : 6$	(4)
Prop. VIII,	$2x : 2y = 48 : 36$	(5)
Prop. IX,	$x : y = 4 : 3$	(6)
From (1), by Prop. VI,	$4 : 3 = x + y : 42$	(7)
From (2), by Prop. VI,	$4 : 3 = x - y : 6$	(8)
From (7), by Prop. I,	$x + y = 56$	(9)
From (8), by Prop. I,	$x - y = 8$	(10)
Hence,	$x = 32$	
And	$y = 24$	

2. Divide the number 14 into two such parts, that the quotient of the greater divided by the less, shall be to the less divided by the greater, as 100 to 16.

SOLUTION.

Let $x =$ the greater;

$y =$ the less

By the conditions, $\begin{cases} \dfrac{x}{y} : \dfrac{y}{x} = 100 : 16 & (1) \\ x + y = 14 & (2) \end{cases}$

From (1), by Prop. IX, $x^2 : y^2 = 100 : 16$ (3)

Prop. XI, $x : y = 10 : 4$ (4)

Hence, from (4), $2x = 5y$ (5)

But $x + y = 14$ (6)

Therefore, $x = 10$

$y = 4$

3. Find three numbers, in geometrical progression, whose sum is 13, and the sum of the extremes is to the double of the mean as 10 to 6.

SOLUTION.

Let x, xy, xy^2, represent the numbers.

By the conditions, $\begin{cases} x + xy + xy^2 = 13 & (1) \\ xy^2 + x : 2xy = 10 : 6 & (2) \end{cases}$

From (2), by Prop. IX, $y^2 + 1 : 2y = 10 : 6$ (3)

From (3), by Pr. VIII, $(y^2 + 2y + 1) : (y^2 - 2y + 1) = 16 : 4$ (4)

Prop. XI, $y + 1 : y - 1 = 4 : 2$ (5)

And $2y : 2 = 6 : 2$ (6)

Hence, $y = 3$

And $x = 1$

4. The product of two numbers is 35, and the difference of their cubes is to the cube of their difference as 109 to 4; what are the numbers?

SOLUTION.

Let x = the greater, and y = the less.

By the conditions, $\begin{cases} xy = 35 & (1) \\ x^3 - y^3 : (x - y)^3 = 109 : 4 & (2) \end{cases}$

Dividing 1st couplet of (2) by $x - y$, $\quad x^2 + xy + y^2 : (x - y)^2 = 109 : 4 \quad (3)$

Or, $\quad x^2 + xy + y^2 : x^2 - 2xy + y^2 = 109 : 4 \quad (4)$

From (4) Prop. VII, $\quad 3xy : x^2 - 2xy + y^2 = 105 : 4 \quad (5)$

But, (1) $\quad 3xy = 105$

Hence, from (5) $\quad (x - y)^2 = 4 \quad (6)$

And $\quad x - y = 2 \quad (7)$

From (1) and (7), $\quad x + y = 12 \quad (8)$

Hence, $\quad x = 7$

And $\quad y = 5$

5. What two numbers are those, whose difference is to their sum as 2 to 9, and whose sum is to their product as 18 to 77?

Ans. 11 and 7.

6. Two numbers have such a relation to each other, that if 4 be added to each, the sums will be to each other as 3 to 4; and if 4 be subtracted from each, the remainders will be to each other as 1 to 4; what are the numbers?

Ans. 5 and 8.

7. Divide the number 16 into two such parts, that their product shall be to the sum of their squares as 15 to 34.

Ans. 10 and 6.

8. There are two numbers whose product is 320, and the difference of their cubes is to the cube of their difference as 61 is to 1; what are the numbers?

Ans. 20 and 16.

APPROXIMATE ROOTS

OF HIGHER DEGREES.

277. A root of any number is evidently one of the equal factors which compose the number. Thus, the square root of 25 is one of the two equal factors, 5×5, which produce 25. The fifth root of 32 is one of the five equal factors, $2 \times 2 \times 2 \times 2 \times 2$, which produce 32.

If a number is not composed of as many equal factors as there are units in the index of the required root, the number is a *surd*, and the required root can only be obtained *approximately*. This may be done, as we have seen (**195**), by extending the general method of extracting the root, to decimal periods. Another method, and the one we are now about to consider, is to decompose the number into factors nearly equal, and *average the result.*

278. If the number a be composed of 3 factors, each equal to x, then

$$\frac{x + x + x}{3} = x = \sqrt[3]{a}.$$

In like manner, if the number a be composed of three factors, x, y, z, *nearly* equal to each other, it is obvious that

$$\frac{x + y + z}{3} = \sqrt[3]{a}, \textit{ nearly.}$$

To illustrate this principle by numeral examples, we have

$$2 \times 3 \times 4 = 24$$

And $$\frac{2 + 3 + 4}{3} = 3$$

But $$\sqrt[3]{24} = 2.88 +.$$

That is, one third of the sum of the three unequal factors, 2, 3 and 4, which compose 24, is 3; while the cube root of 24 is 2 88 +, a number less than 3 by only .12.

Again, take $5 \times 7 \times 8 = 280$

And $\dfrac{5 + 7 + 8}{3} = 6.66 +$

But $\sqrt[3]{280} = 6.54 +$

Difference, $.12 +.$

Similarly, we shall find that if $uxyz = a$,

Then $$\frac{u + x + y + z}{4} = \sqrt[4]{a}, \text{ nearly.}$$

Hence, in general terms,

If a number be decomposed into n factors, nearly equal to each other, the nth part of their sum will be nearly equal to the nth root of the number.

279. If a number does not consist of factors nearly equal, or if it is a prime number, it may be decomposed into *approximate* factors.

Thus, if $xyz = a$,

Then $x = \dfrac{a}{yz}$,

in which, if y and z be *assumed*, x may be *computed.*

1. Find the cube root of 100.

OPERATION.

Let $xyz = 100$

Assume $y = 4$

And $z = 5$

Then $x = \dfrac{100}{yz} = \dfrac{100}{20} = 5$

Adding values of $x+y+z$, $4+5+5 = 14$

Hence, dividing by 3, $\sqrt[3]{100} = 4.66+$, 1st approx.

As the root sought is greater than 4.6, and less than 4.7, it must evidently be some number between these two. We will therefore

Next, assume $y = 4.6$

And, $z = 4.7$

Then, $x = \frac{100}{yz} = \frac{100}{21.62} = 4.62503+$

Adding the new values, $x + y + z = 13.92503+$

Hence, dividing by 3, $\sqrt[3]{100} = 4.64167+$, 2d approx.

Next, assume $y = 4.641$

And, $z = 4.642$

Then, dividing as before, $x = 4.64176+$

Adding the new values, $x + y + z = 13.92476+$

Dividing by 3, $\sqrt[3]{100} = 4.64158+$, 3d approx.

which is correct to the last decimal place.

2 Find the square root of 3.

OPERATION.

Let $xy = 3$

Assume $y = 1.6$

Then, $x = \frac{3}{1.6} = 1.875 +$

Adding, $x + y = 3.475 +$

Dividing by 2, $\sqrt{3} = 1.7375$, 1st approximation.

Next, assume $y = 1.732$

Then, $x = \frac{3}{1.732} = 1.7321016 +$

Adding, $x + y = 3.4641016 +$

Dividing by 2, $\sqrt{3} = 1.7320508+$, 2d approx.

which is correct to the last decimal place.

Hence, to obtain the approximate nth root of a number, we have the following

RULE. I. *Assume $n - 1$ factors as near the required root as may be found by inspection.*

II. *Divide the given number by the product of the assumed factors, and the quotient will be the remaining factor.*

III. *Divide the sum of the n factors thus obtained, by n, and the quotient will be the first approximation to the required root.*

IV. *Assume a second set of $n - 1$ factors, and proceed as before; and thus continue, till the desired approximation is obtained.*

NOTE. 1. By inspecting the two examples given, it will be seen that the second approximation is less than the first, and the third less than the second. Hence, every approximation is greater than the true root, and this principle will govern the selection of the factors to be assumed in each step.

2. Also, by inspecting the two preceding operations it will be seen that each approximation after the first obtains one or more correct decimal figures in the required root.

EXAMPLES FOR PRACTICE.

3. Required the 4th root of 18.

OPERATION.

Let $uxyz = 18$

Assume, $\begin{cases} u = 2 \\ x = 2 \\ y = 2 \end{cases}$

Then by (II), $z = 2.25$

Adding, $u + x + y + z = 8.25$

Dividing by 4 (III), $\sqrt[4]{18} = 2.0625$, 1st approx.

Next, assume $\begin{cases} u = 2.06 \\ x = 2.06 \\ y = 2.05 \end{cases}$

Then, as before, $z = 2.069113 +$

Adding, $u + x + y + z = 8.239113 +$

Dividing by 4, $\sqrt[4]{18} = 2.059778 +$, 2d approx.

NOTE.—It will be observed, in the above example, that 16, a number near 18, is a perfect 4th power, whose root is 2. Hence, u, x and y were assumed each as 2.

4. Required the cube root of 130.
5. Required the 4th root of 260.
6. Required the 4th root of 640.
7. Required the 5th root of 2.
8. Required the 5th root of 38.

280. When the numbers are large, or the index of the root is higher than the 3d or 4th, the following table will be of use in assuming the factors for the 1st approximation. It will be seen that the first column contains certain numbers, taken at intervals between 1 and 27000; the second column contains the squares of the numbers in the third; the third column contains the cube roots of the numbers in the first; the fourth contains the 6th roots; and the fifth, the 9th roots.

A	$A^{\frac{2}{3}}$	$A^{\frac{1}{3}}$	$A^{\frac{1}{6}}$	$A^{\frac{1}{9}}$
1	1	1	1.0000000	1.000000
8	4	2	1.4142136	1.259921
27	9	3	1.7320508	1.442250
64	16	4	2.0000000	1.587401
125	25	5	2.2360680	1.709976
216	36	6	2.4494897	1.817121
343	49	7	2.6457513	1.912933
512	64	8	2.8284271	2.000000
729	81	9	3.0000000	2.080084
1000	100	10	3.1622777	2.154435
1331	121	11	3.3166248	2.223980
1728	144	12	3.4641016	2.289428
2197	169	13	3.6055513	2.351335
2744	196	14	3.7416574	2 410142
3375	225	15	3.8729833	2.466212
4096	256	16	4.0000000	2.519842
4913	289	17	4.1231056	2.571282
5832	324	18	4.2426407	2.620741
6859	361	19	4.3588989	2.668402
8000	400	20	4.4721360	2.714418
9261	441	21	4.5825757	2.758923
10648	484	22	4.6904158	2.802039
12167	529	23	4.7958315	2.843867
13824	576	24	4.8989795	2.884499
15625	625	25	5.0000000	2.924018
17576	676	26	5.0990195	2.962496
19683	729	27	5.1961524	3.000000
21952	784	28	5.2915026	3.036589
24389	841	29	5.3851648	3.072317
27000	*900*	30	5.4772256	3.107232

1. Find the 7th root of 2211.

Referring this number to the table, we find the number nearest to it in column A is 2197, whose 6th root ($A^{\frac{1}{6}}$), is is 3.6 +, and whose 9th root ($A^{\frac{1}{9}}$), is 2.3 +. Therefore, the 7th root of 2211 must be nearly equal to 3, and we will take this number for each of the six factors to be assumed. Thus,

OPERATION.

Let $$3 \times 3 \times 3 \times 3 \times 3 \times 3x = 2211$$

$$\begin{array}{r|r} 3 & 2211 \\ \hline 3 & 737 \\ \hline 3 & 245.666666 \\ \hline 3 & 81.888888 \\ \hline 3 & 27.296296 \\ \hline 3 & 9.098765 \\ \hline & 3.032921 = x. \end{array}$$

Hence, $$\sqrt[7]{2211} = \frac{3 + 3 + 3 + 3 + 3 + 3 + 3.032921}{7}$$

$$= \frac{21.032921}{7}$$

$$= 3.004703, \text{ 1st approximation.}$$

For the second approximation, we may assume 3.004 for each of the 6 equal factors. As the method above used would be somewhat tedious, we may avoid the labor, by an application of the binomial theorem.

For this purpose let it be observed that the powers of a fraction are less than the fraction itself. Thus,

$$(.1)^2 = .01$$
$$(.1)^3 = .001$$
$$(.1)^4 = .0001$$

We have found that the 7th root of 2211 is greater than 3, by a small fraction. Let this fraction be represented by x; and we shall have the following equation:

$$(3 + x)^7 = 2211.$$

Expanding the first member, indicating the powers of 3, we have

$$2187 + 7(3)^6x + 21(3)^5x^2 + 35(3)^4x^3 + 35(3)^3x^4 + 21(3)^2x^5 + 7(3)x^6 + x^7 = 2211.$$

Now as x is a small fraction, the powers of x will be still smaller. And if the terms containing x^3, and all the higher powers of x be omitted, the equation will still be approximately true; and we shall have

$$2187 + 7(3)^6x + 21(3)^5x^2 = 2211 \quad (1)$$

Transposing, $7(3)^6x + 21(3)^5x^2 = 24$ (2)

Dividing (2) by 3, $7(3)^5x + 21(3)^4x^2 = 8$ (3)

Expanding (3), $1701x + 1701x^2 = 8$ (4)

Dividing (4) by 1701, $x^2 + x = .0047031$ (5)

Completing the square, $4x^2 + (\ ) + 1 = 1.0188124$ (6)

Extracting square root, $2x + 1 = 1.00936$ (7)

Reducing (7), $x = .00468$

Hence, $\sqrt[7]{2211} = 3 + x = 3.00468$, 2d approx.

2. Find the 7th root of 2412.

Referring to the table, we find this number in column A, between 2197 and 2744; and by examining the 6th and 9th roots of these numbers, we find that the 7th root of the given number must be a little greater, or a little less, than 3. Hence,

OPERATION.

Let $3 + x =$ required root.

Put $a = 3$

Then, $(a + x)^7 = 2412$

Expanding, $a^7 + 7a^6x + 21a^5x^2 + 35a^4x^3 + \&c. = 2412$

Or, approximately, $a^7 + 7a^6x = 2412$ (1)

Restoring value of a in (1), $2187 + 5103x = 2412$ (2)

Transposing, $5103x = 225$ (3)

And $x = .044 +$

Hence, $3 + x = 3.044 +$

That is, the seventh root of 2412 is very nearly 3.044; but this is a little too large. If we would find the root more accurately we will place 3.04 for a, or $(3 + m)$ in equation (1). Then that equation will become

$$(3 + m)^7 + 7\,(3 + m)^6\, x = 2412.$$

Taking the higher terms of $(3 + m)^7$, and $(3 + m)^6$ in separate columns, we have

$$\begin{array}{rlrl} 3^7 = & 2187 & 3^6 = & 729 \\ 7(3)^6 m = & 204.12 & 6(3)^5 m = & 58.32 \\ 21(3)^5 m^2 = & 8.1648 & 15(3)^4 m^2 = & 1.944 \\ 35(3)^4 m^3 = & .18144 & 20(3)^3 m^3 = & .03456 \\ 35(3)^3 m^4 = & .0024192 & 15(3)^2 m^4 = & .0003456 \\ \hline & & & 789.2989056 \end{array}$$

Adding, $2399.4686592 + 7(789.2989056)x = 2412$

Whence, from the above equation, we have approximately

$$2399.4686592 + 7(789.2989056)x = 2412\,;$$

Or, $x = \dfrac{12.5313408}{7(789.2989056)} = .002268 +;$

Hence, $\sqrt[7]{2412} = 3.042268 +.$

EXAMPLES FOR PRACTICE.

3. What is the cube root of 14000?
4. What is the 5th root of 812? *Ans.* 3.81893.
5. What is the 8th root of 1340?
6. What is the 7th root of 9150?

MISCELLANEOUS EXAMPLES.

1. What is the value of $\dfrac{a\sqrt{b + 2\sqrt{mb}}}{m\sqrt{b - (1 - m)a}}$, when $a = 4$, $b = 5$, and $m = 20$?

Ans. $\frac{1}{9}$.

2. What is the sum of $(a + b)\sqrt{ax} + 12(a + b)\sqrt{ax} - 7(a + b)\sqrt{ax}\ 4n(a + b)\sqrt{ax}$?

Ans. $\Big(6(a + b) + 4n(a + b)\Big)\sqrt{ax}$;

Or, $(6 + 4n)(a + b)\sqrt{ax}$.

3. What is the sum of $(A + B)^2(x + y)$ and $(A - B)^2(x + y)$?

Ans. $2(A^2 + B^2)(x + y)$.

4. What is the sum of $5a^4b + 3a^{-2}b^2c$, $6a^4b + 2a^{-2}b^2c$, $+ 10\,ab$, and $9a^4b - 8a^{-2}b^2c - 10ab$?

Ans. $20a^4b - \dfrac{3b^2c}{a^2}$.

5. From $\sqrt{x^2 - y^2} + 4(x + y) - 3\sqrt{a + x}$, subtract $3(x + y) - 2(x^2 - y^2)^{\frac{1}{2}} + 13(a + x)^{\frac{1}{2}}$.

Ans. $3\sqrt{x^2 - y^2} + x + y - 16\sqrt{a + x}$.

6. From $17ax^2 + 3ay + 10a$, subtract $7ax^2 + 4ay + 12a - 2ba$.

Ans. $(10x^2 - y - 2 + 2b)a$.

7. What is the value of $(2x)(-2xy)(-2mx)(-2xy^3)(2m^4xy)$?

Ans. $-(2mxy)^5$

8. Multiply $a^3 - a^2b + ab^2 - b^3$ by $a + b$.

Ans. $a^4 - b^4$.

9 Multiply $x^m - x^{m-1}y + x^{m-2}y^2 - y^m$ by $x + y$.

Ans. $x^{m+1} - xy^m + x^{m-2}y^3 - y^{m+1}$

10. Expand $(m - z - 1)(m + 1)(m^2 + 1)$.
Ans. $m^4 - m^3z - m^2z - mz - z - 1$.

11. Expand $(3x^2y - 3xy^2)(3x^2y - 3xy^2)$.
Ans. $9x^4y^2 - 18x^3y^3 + 9x^2y^4$.

12. Expand $(c^m + c^n)(c^m + c^n)$.
Ans. $c^{2m} + 2c^{m+n} + c^{2n}$.

13. Find two factors of $x^5 + y^5$.
Ans. $(x + y)$ and $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

14. Find two factors of $x^5 - y^5$.
Ans. $\left(x^{\frac{5}{2}} + y^{\frac{5}{2}}\right)$ and $\left(x^{\frac{5}{2}} - y^{\frac{5}{2}}\right)$;
Or, $(x - y)$ and $x^4 + x^3y + x^2y^2 + xy^3 + y^4$.

15. Multiply $\left(a^{\frac{5}{3}} - a^{\frac{4}{3}}b + ab^2 - a^{\frac{2}{3}}b^3 + a^{\frac{1}{3}}b^4 - b^5\right)$ by $\left(a^{\frac{1}{3}} + b\right)$. *Prod.* $a^2 - b^6$.

16. Find the factors of $a^8 - b^4$.
Ans. $(a^2 - b)(a^6 + a^4b + a^2b^2 + b^3)$, or $(a^4 + b^2)(a^4 - b^2)$.

17. Find the greatest common factor of $(a^4 - 1)$, $(a^5 + a^3)$, $(a^6 + 1)$. *Ans.* $a^2 + 1$.

18. Find the greatest common divisor of $a^2 - 5ab + 4b^2$, and $a^2 - ab + 3ab^2 - 3ab^3$. *Ans.* $a - b$.

19. Find the sum of the following fractions; indicating that sum by S, and condensing it to a single term.

$$\frac{a^2}{(a+b)^2},\ \frac{b}{a+b},\ \text{and}\ \frac{ab}{(a+b)^2}. \qquad \textit{Ans. } 1.$$

20. What is the sum of the following fractions?

$$\frac{x}{x^2 - y^2},\ \frac{y}{x + y},\ \text{and}\ \frac{1}{x - y}. \qquad \textit{Ans. } \frac{2x + xy - y^2 + y}{x^2 - y^2}.$$

21. Divide $51a^2(c - 1)^{m+n}$ by $17a^2(c - 1)^{m-n}$.
Ans. $3(c - 1)^{2n}$.

22. Divide $5(a - 1) + 6(1 - a)^2$ by $1 - a$.
Ans. $1 - 6a$.

23. Divide $m - n$ by $\sqrt{m} - \sqrt{n}$. *Ans.* $\sqrt{m} + \sqrt{n}$.

24. Divide $a^2 + ab + b^2$ by $a + \sqrt{ab} + b$.

Ans. $a - \sqrt{ab} + b$

25. Factor $x^2y - xy^2 - xy + x - y$.

Ans. $(x-y)\left((xy+1) - \frac{xy}{x-y}\right)$.

26 What is the greatest common divisor of $4a^3z - 12amz^2 + 8abcz$, and $5a^2mx - 15m^2xz + 10bcmx$?

Ans. $a^2 - 3mz + 2bc$.

27. What is the least common multiple of $3a^2x$, $4axy$, and $16ax^3$? *Ans.* $48a^2x^3y$.

28. What is the least common multiple of $ac - 3m^2c$, $a^2c + 3acm^2$ and $a^2c - 9cm^4$? *Ans.* $a^3c - 9acm^4$.

29. Reduce $\frac{\sqrt{a^3b} - \sqrt{ab^3}}{a^2 - 2ab + b^2}$ to its lowest terms.

Ans. $\frac{\sqrt{ab}}{a-b}$.

30. Reduce $\frac{m(\sqrt{a} - \sqrt{b})}{(\sqrt{a} - \sqrt{b})^{-1}}$ to an entire form.

Ans. $m(a + b) - 2m\sqrt{ab}$.

31 Find the value of $\frac{c}{c-1} - \left(\frac{c^2}{c^2-1} + \frac{2c}{c^4-1}\right)$.

Ans. $\frac{c}{c^2+1}$.

32. From $\frac{1}{x^3} + \frac{1}{x^2} + \frac{x-1}{x^2+1}$, subtract $\frac{1}{x} + \frac{1}{(x^2+1)}$.

Diff. $\frac{1+x-x^3}{x^3(x^2+1)}$.

33. Divide $1 + \frac{n-1}{n+1}$, by $1 - \frac{n-1}{n+1}$. *Ans.* n.

Note.—The product of the divisor and quotient must be equal to the dividend. Let $Q =$ the quotient, then

$$Q\left(1 - \frac{n-1}{n+1}\right) = 1 + \frac{n-1}{n+1}.$$

84. Divide $\dfrac{a^2x^2}{a^2-x^2}$, by $\left(x - \dfrac{ax}{a+x}\right)$. *Ans.* $\dfrac{a^2}{a-x}$.

85. Simplify the fraction, $\dfrac{\frac{a+x}{a-x}+\frac{a-x}{a+x}}{\frac{a+x}{a-x}-\frac{a-x}{a+x}}$. *Ans.* $\dfrac{a^2+x^2}{2ax}$.

86. Divide $2 - \dfrac{\sqrt{a}}{\sqrt{b}} - \dfrac{\sqrt{b}}{\sqrt{a}}$, by $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{ab}}$.

Ans. $\sqrt{b} - \sqrt{a}$.

87. What is the value of $\dfrac{(a+b)^2-(a-b)^2}{(a^2-b^2)-(a-b)^2}$?

Ans. $\dfrac{2a}{a-b}$.

88. What is the value of $\dfrac{(a+b)^3-(a-b)^3}{(1+b)^3-(1-b)^3}$?

Ans. $\dfrac{3a^2+b^2}{3+b^2}$.

89. Raise $\sqrt{a+b}$ to the 4th power. *Ans.* $(a+b)^2$.

40. Raise $a\sqrt{-1}$ to the 3d power. *Ans.* $a^3\sqrt{-1}$.

41. Raise $2a+3b$ to the 9th power.

Ans. $512a^9 + 256 \cdot 27a^8b + 512 \cdot 81a^7b^2 + 256 \cdot 567a^6b^3 +$ &c.

Reduce the following equations:

42. $(x+\sqrt{a})^2 - (x-\sqrt{a})^2 = 2a\sqrt{ab}$.

Ans. $x = \dfrac{a\sqrt{b}}{2}$.

43. $\dfrac{x\sqrt{3am}}{m} = \sqrt{3a} + \dfrac{\sqrt{12ac^2}}{m}$. *Ans.* $x = \dfrac{m+2c}{\sqrt{m}}$.

44. $\dfrac{b}{x} + \dfrac{\sqrt{a}}{x} = \dfrac{b+\sqrt{a}}{\sqrt{ab^2}}$. *Ans.* $x = b\sqrt{a}$.

45. $\dfrac{4}{x-1} - \dfrac{x+1}{4} = \dfrac{x+1}{4}$. *Ans.* $x = \pm 3$.

46. $\dfrac{5x^2+5}{5x+5} = 5x - 5$. *Ans.* $x = \pm 1.224744 +$.

47. $\dfrac{x}{x^2-1} = \dfrac{x^2+1}{x}$. *Ans.* $x = \pm\frac{1}{2}\sqrt{2 \pm 2\sqrt{5}}$.

48. $\sqrt{5x-9} = \dfrac{4x}{\sqrt{5x-9}}$. *Ans.* $x = 9$.

49 $(1+x)^6 + (1-x)^6 = 242$.
Ans. $x = \pm 2$ or $\pm \sqrt{-6}$.

50. Find the 6th power of $\sqrt{2} + \sqrt{3}$, or expand $(\sqrt{2x} + \sqrt{3y})^6$, and afterward suppress all the powers of x and y, on the supposition that each is equal to unity.
Ans. $8 + 24\sqrt{6} + 180 + 120\sqrt{6} + 270 + 54\sqrt{6} + 27$.

51. Given $\dfrac{y^2-9}{y+3} = 7 - y$, to find the value of y.
Ans. $y = 5$.

52. Given $\dfrac{25x^2-9}{5x+3} = 7 - 5x$, to find the value of x.
Ans. $x = 1$.

53. Given $\dfrac{x}{2} + 2 = \dfrac{x}{2}\left(\dfrac{6}{x} - 1\right) + \dfrac{7\frac{1}{2} - \frac{1}{2}x}{3}$, to find the value of x. *Ans.* $x = 3$.

54. Given $x^3 + 2x^2 + x = (x^2 + 3x)(x - 1) + 32$, to find the value of x. *Ans.* $x = 8$.

55. Given $(x+1)^6 - (x-1)^6 = 1344x$.
Ans. $x = \pm 3$, or $\pm \sqrt{-\frac{37}{3}}$.

56. $\begin{cases} 5x + y = 26 \\ x + 5y = 10 \end{cases}$ *Ans.* $\begin{cases} x = 5. \\ y = 1. \end{cases}$

57. $\begin{cases} ax + cy = 2a^2c^2 \\ cx + ay = ac(c^2 + a^2) \end{cases}$ *Ans.* $\begin{cases} x = ac^2. \\ y = a^2c. \end{cases}$

58. $\begin{cases} x + y - z = 4 \\ 2x + 2z - y = 6 \\ 3y + 3z - x = 6 \end{cases}$ *Ans.* $\begin{cases} x = 3. \\ y = 2. \\ z = 1. \end{cases}$

59. $\begin{cases} u + x + y = 3a - 6m \\ u + x + z = 3a - 7m \\ u + y + z = 3a - 8m \\ x + y + z = 3a - 9m \end{cases}$ *Ans.* $\begin{cases} u = a - m. \\ x = a - 2m. \\ y = a - 3m. \\ z = a - 4m. \end{cases}$

60. $\begin{cases} x + 2y = 7 \\ 4x^2 - 4y = 28 \end{cases}$ *Ans.* $\begin{cases} x = 3, \text{ or } -3\frac{1}{2}. \\ y = 2, \text{ or } \quad 5\frac{1}{4}. \end{cases}$

61. $\begin{cases} x^2 + 3y^2 = 28 \\ x^2 + 2xy = 35 \end{cases}$ *Ans.* $\begin{cases} x = 5, \text{ or } \sqrt{21}. \\ y = 1, \text{ or } \frac{1}{3}\sqrt{21}. \end{cases}$

62. $\begin{cases} x + \sqrt{xy} = 15 \\ y + \sqrt{xy} = 10 \end{cases}$ *Ans.* $\begin{cases} x = 9. \\ y = 4. \end{cases}$

63. $\begin{cases} x^4 + y^2 = 106 \\ x^2y = 45 \end{cases}$ *Ans.* $\begin{cases} x = 3, \text{ or } 3\sqrt{-1}. \\ y = 5, \text{ or } \quad -5. \end{cases}$

64. Two men started from two towns, A, and B, and traveled toward each other. The first went $\frac{1}{3}$, and the second, $\frac{2}{5}$ of the distance between the two towns, when the men were found to be 16 miles apart; required the distance from A to B.

Ans. 60 miles.

65. The sum of two numbers is 80; and if their difference be subtracted from the less and added to the greater, the results will be as 1 to 7; what are the numbers?

Ans. 30 and 50.

66. What number is that whose fourth part exceeds its fifth part by $\frac{1}{40}$? *Ans.* $\frac{1}{2}$.

67. There is a number whose 3 digits are the same; and if from the number, 4 times the sum of the digits be subtracted, the remainder will be 297; required the number.

Ans. 333.

68. A certain number increased by 1, is to the same number increased by 4, as the square of the number is to its cube; what is the number? *Ans.* 2

69. Fifty gallons of wine are to be put into casks of two sizes; and 10 of the smaller casks and 2 of the larger, or 5 of the smaller and 6 of the larger, may be used; required the capacity of a cask of each size.

Ans. Smaller, 4 gal.; larger, 5 gal.

70. Two men have the same income; one saves one tenth of his, the other spends $150 per annum more than the first, and at the end of five years finds himself $100 in debt; what is the income of each? *Ans.* $1300.

71. A man has 2 equal flocks of sheep; from one he sells a sheep, and from the other b sheep, and then he has 3 times as many remaining in the latter flock as in the former; how many did each flock originally contain? *Ans.* $\frac{1}{2}(3a - b)$ sheep.

NOTE.—For a and b, take any number at pleasure, such that $3a - b$ shall be divisible by 2, and form a definite problem. For instance, assume $a = 12$, and $b = 2$, then the number in each flock will be 17.

In this manner numeral problems are formed.

72. The sum of two numbers is 72, and the sum of their cube roots is 6; what are the numbers? *Ans.* 64 and 8.

73. The sum of the squares of two numbers, multiplied by the sum of the numbers is 2336, and the difference of their squares, multiplied by the difference of the numbers is 576; what are the numbers? *Ans.* 11 and 5.

74. The product of two numbers multiplied by their sum is 84; and the sum of their squares multiplied by the square of their product is 3600; what are the numbers? *Ans.* 4 and 3.

75. A market man bought 15 ducks and 12 turkeys for 105 shillings, and he obtained 2 more ducks for 18 shillings, than turkeys for 20 shillings; what were the prices?

Ans. Ducks, 3 shillings; turkeys, 5 shillings.

76. The sum of three numbers is 12; one third of the sum of the first and second is equal to one fifth of the sum of the second and third; and the second minus the first is equal to the third minus the second; required the numbers?

Ans. 2, 4, and 6.

77. There is a square tract of land containing 10 times as many acres as there are rods in the fence inclosing it; how large is the square? *Ans.* 20 miles square.

78. A general wishing to draw up his regiment into a square, found by trial that he had 92 men over; he then increased each side by 2 men, and wanted 100 men to complete the square; how many soldiers had he? *Ans.* 2301.

79. Some bees had alighted upon a tree; at one flight the square root of half the number went away; at another $\frac{8}{9}$ of them; and two bees then remained; how many alighted upon the tree? *Ans.* 72.

80. A May-pole is 56 feet high. At what distance above the ground must it be broken, in order that the upper part, clinging to the stump, may touch the ground 12 feet from the foot? *Ans.* $26\frac{2}{7}$ feet.

81. Divide the number 20 into two such parts that the square of the greater diminished by twice the less, shall be equal to twice the square of the less. *Ans.* 12 and 8.

82. Three numbers are in arithmetical progression; their sum is 27, and the product of the extremes is 77; required the numbers. *Ans.* 7, 9, and 11.

83. A gentleman has a garden 10 rods long and 8 rods wide; he would lay out half the way round it a graveled walk of uniform width and to contain $\frac{1}{6}$ of the area of the garden. How wide shall the walk be laid out? *Ans.* 13.4376+ft.

84. Two numbers are in the proportion of a to b, and when c is added to each, the proportion is as 5 to 6; what are the numbers? *Ans.* $\frac{ac}{5b-6a}$, and $\frac{bc}{5b-6a}$.

85. A man sold a horse for 144 dollars, and gained as much per cent as the horse cost him. What did the horse cost him? *Ans.* $80.

86. Two numbers are in the proportion of 5 to 8, and if 200 be added to the first, and 120 to the second, the sums will be to each other as 5 to 4; what are the numbers?

Ans. 50 and 80.

87. A person bought two cubical stacks of hay for £41, each of which cost as many shillings per cubic yard, as there were yards in a side of the other, and the greater stood on more ground than the less by 9 square yards. What was the value of each stack?

Ans. £16 and £25.

Let x = a side of the greater stack in yards;

And y = a side of the other;

Then $x^2 - y^2 = 9 = a$; (1)

And $x^2y + xy^2 = 41 \cdot 20 = 820 = b$ (2)

From (2) $x^2 + y^2 = \dfrac{b}{xy}$ (3)

Squaring (3) $x^4 + 2x^2y^2 + y^4 = \dfrac{b^2}{x^2y^2}$ (4)

Squaring (1) $x^4 - 2x^2y^2 + y^4 = a^2$

Diff $4x^2y^2 = \dfrac{b^2}{x^2y^2} - a^2$.

88. A farmer had 3 more cows than horses. He bought 2 more cows and sold 3 horses; and he then had 5 times as many cows as horses. How many had he at first?

Ans. 5 horses and 8 cows.

89. Some boys on a frolic incurred a bill of $12. If there had been two more in the company each would have been charged 30 cents less. How many were in company? *Ans.* 8.

90. A person residing on the bank of the Ohio, 15 miles above Cincinnati, can row his boat to the city in $2\frac{1}{2}$ hours, but it requires $7\frac{1}{2}$ hours to return. With what force can he row his boat in still water, and what is the velocity of the river?

Ans. Man rows 4 miles per hour; stream flows 2 miles.

91. What is the value of $x + \dfrac{2x}{x-3}$ divided by $x - \dfrac{2x}{x-3}$ when $x = 5\frac{1}{2}$?

Ans. 9.

92. I deposited $200 in a savings bank, which paid 6 per cent. on deposits, interest payable semi-annually. How much did my money amount to in 5 years, the interest being added to the principal at the end of every 6 months?

NOTE.—The principal is the first term; $1.03, the semi-annual amount of $1 at the rate per cent. is the ratio; and the number of terms minus 1 is 2 times the number of years.

Ans. $268.78+.

93. What is the value of the expression $\frac{x+2a}{x-2a}+\frac{x+2b}{x-2b}$ when $x=\frac{4ab}{a+b}$. *Ans.* 2.

94. Find the geometrical mean between $2a^2b^{\frac{1}{3}}$ and $24a^3b^{\frac{1}{2}}x^{\frac{1}{4}}$.

Ans. $4a^2\sqrt{3a}\times\sqrt[12]{b^5}\times\sqrt[8]{x}$.

95. From a bag of money which contained a certain sum, was taken $20 less than its half; from the remainder, $30 less than its third part; and from the remainder, $40 less than its fourth part, and then there was nothing left. What sum did the bag contain? *Ans.* 1080.

96. Four numbers are in arithmetical progression; the product of the first and third is 27, and the product of the second and fourth is 72. What are the numbers?

Ans. 3, 6, 9, and 12.

97. A merchant gains the first year, 15 per cent. on his capital; the second year, 20 per cent. on the capital at the close of the first; and the third year, 25 per cent, on the capital at the close of the second; when he finds that he has cleared $1000.50. Required his capital. *Ans.* $1380.

98. The sum of three numbers in arithmetical progression is 15, and their product is 80. Required the numbers.

Ans. 2, 5, and 8.

99. Find three numbers in arithmetical progression such that the sum of their squares shall be 2900, and the product of the extremes shall be less than the square of the mean by 100. Ans. 20, 30, and 40.

100. What number is that which, if 4 be subtracted from it, $\frac{1}{3}$ of the remainder will be 7? *Ans.* 25.

101. What number is that to which if 1 and 11 be added separately, the sums will be to each other as 1 to 3?

Ans. 4.

102. Given $\frac{a^3b}{(a+b)^3} + \frac{(2a+b)bx}{(a+b)^2} = x$, to find the value of x.

Ans. $x = \frac{ab}{a+b}$.

103. What number is as much below 40, as three times that number is below 100? *Ans.* 30.

104. Divide 400 into two such parts, that the sum of their square roots shall be 28. *Ans.* 256 and 144.

105. A man sold a horse for a dollars, and gained as much per cent as the horse cost him; what did the horse cost him?

Ans. $(\sqrt{a+25})\,10 - 50$ dollars.

106. The product of three numbers in geometrical progression is 1728, and the sum of the first and third is 40; what are the numbers? *Ans.* 4, 12, and 36.

107. Two quantities are to each other as m to n, and the difference of their square is d^2; what are the quantities?

Ans. $\frac{md}{\sqrt{m^2-n^2}}$, $\frac{nd}{\sqrt{m^2-n^2}}$.

108. The sum of four numbers in geometrical progression is 85; and the sum of the first two is to the sum of the second two as 1 to 16; what are the numbers?

Ans. 1, 4, 16, and 64.

109. The base of a right angled triangle is 20 rods, and the perpendicular and hypotenuse are to each other as 5 to 7; what is the length of the perpendicular and what the area of the triangle?

Ans. Perpendicular, 20.4124 + rods; area, 204.124 + sq. rods.

110. The extremes of a geometrical series are 2 and 3744, and the difference of the first and second term is to the difference of the third and fourth as 1 to 9. Required the sum of the series. *Ans.* 5615.

111. The sum of two numbers added to the sum of their squares is 18, and 10 times their product is 60; what are the numbers? *Ans.* 2 and 3, or $\sqrt{3} - 3$, and $-\sqrt{3} - 3$.

112. The sum of two numbers is to their difference as 4 to 1, and the sum of their cubes is 152; what are the numbers? *Ans.* 3 and 5.

113. Insert 9 arithmetical means between 6 and 36. *Ans.* 9, 12, 15, &c.

114. At what rate per cent. will a dollars gain as much in 4 years at simple interest, as in 2 years at compound interest? *Ans.* 200 per cent.

115. How many terms of the series, .034, .0344, .0348, &c., will amount to 2.748? *Ans.* 60.

116. How much will \$230 amount to in 12 years, at 6 per cent., simple interest?

NOTE.—The number of terms will evidently be 1 greater than the number of years.

Ans. \$395.60.

117. What is the sum of n terms of the series, 3, $3\frac{1}{3}$, $3\frac{2}{3}$, &c.? *Ans.* $(n + 17)\frac{n}{6}$.

118. I lent a certain sum at 7 per cent., simple interest, and at the end of 5 years received, in principal and interest, \$317.79; what was the sum lent? *Ans.* \$235.40.

119. If 6 be the first term of a geometrical series, and 4374 the 7th term, what is the ratio, and what are the other terms?

Ans. Ratio 3; whence 18, 54, &c., the other terms of the series.

120. Sold a horse for \$175.50, taking a note drawing interest at 6 per cent. Not needing the money, I did not collect the note until the end of 6 years; what amount did I collect? *Ans.* \$238.68.

121. The sum of the extremes of 4 numbers in geometrical progression is 35, and the sum of the means is 30; what are the numbers? *Ans.* 8, 12, 18, 27.

122. I own a mortgage of \$875 on a farm, due in 6 years, at 6 per cent. interest, payable annually. If no part of the mortgage or interest is paid until the end of the 6 years, how much will be the amount due at compound interest? *Ans.* \$1241.20 +.

123. Three times the product of two numbers is equal to their difference multiplied by the difference of their squares. Also, 45 times the square of the product is equal to the difference of their 4th powers multiplied by the difference of their squares; what are the numbers? *Ans.* 4 and 2.

124. The principal is \$300, the time 3 years, and the rate 6 per cent., compound interest, semi-annually; what is the amount?

NOTE. — The number of terms is 1 more than 2 times the number of years; and the ratio is 1.03.

Ans. \$358.2156.

125. The length of a plat of ground is 4 rods more than its breadth; and the number of square rods in its area is equal to the number of rods in its perimeter. Required the length and breadth.

$$Ans. \begin{cases} \text{Length, } 6.8284 + \text{ rods.} \\ \text{Breadth, } 2.8284 + \text{ rods.} \end{cases}$$

126. Find the side of a cube which shall contain as many solid units as there are linear units in the distance between its two opposite corners. *Ans.* $\sqrt[4]{3}$.

127. Find two numbers, such that their product shall be equal to 4 times their difference; and the difference of their squares shall be 9 times the sum of the numbers.

Ans. 3 and 12.

128. A and B together carried 90 eggs to market, and sold at different prices, each receiving the same sum. Had A taken as many as B he would have received 32 cents for them. Had B taken as many as A, he would have received 50 cents for them; how many did each take to market?

Ans. A 50, B 40.

129. Find two numbers, such that the difference of their squares shall be 12; and 3 times the square of the greater minus twice their product shall be 32. *Ans.* 4 and 2.

130. The product of 5 numbers in arithmetical progression is 945, and the sum of the numbers is 25; what are the numbers? *Ans.* 1, 3, 5, 7, 9.

131. A man has two unequal measures. If he lay out a plat of ground having the greater measure for its length, and the less for its breadth, it will contain 40 square feet; but if he lay out a plat having twice the greater measure and once the less for its length, and once the greater and twice the less for its breadth, it will contain 432 square feet. How many feet in length is each measure? *Ans.* Less, 4; greater, 10.

132. A carpenter agreed to live with a farmer during the winter, on the condition that for every day he worked he should receive $1.50, and for every day he was idle he should forfeit 65 cents. At the expiration of 129 days they settled, and the carpenter received nothing; how many days did he work, and how many was he idle?

Ans. He worked 39 days, and was idle 90 days.

133 Find three numbers such that the product of the first and second shall be 6; of the first and third, 8; and of the second and third, 12. *Ans.* 2, 3, and 4

134. In a plane triangle the base is 50 feet, the area 600 feet, and the difference of the sides 10 feet; required the sides and perpendicular. *Ans.* Sides, 30 and 40; perpen. 24 feet.

135. There are four numbers such that the product of the first, second, and third is a; the product of the first, second, and fourth is b; the product of the first, third, and fourth is c; and the product of the second, third, and fourth is d. Required the numbers.

$$Ans.\ \frac{\sqrt[3]{abcd}}{d},\ \frac{\sqrt[3]{abcd}}{c},\ \frac{\sqrt[3]{abcd}}{b},\ \frac{\sqrt[3]{abcd}}{a}.$$

136. A is a traveler 11 miles in advance of B, and travels 4 miles per hour; B starts to overtake him, and travels $4\frac{1}{2}$ miles the first hour, $4\frac{3}{4}$ the second, and 5 the third, increasing his rate $\frac{1}{4}$ of a mile per hour; how many hours before he will overtake A? *Ans.* 8.

137. The base of a right angled triangle is 6, and the three sides are in arithmetical progression; what are the sides?

Ans. 6, 8, 10, or $4\frac{1}{2}$, 6, $7\frac{1}{2}$.

138. Two squares contain together an area of 52 inches; and twice the difference of their area is equal to the number of inches in the perimeters of the two. Required the contents of each.

Ans. Greater, 36 inches; Less, 16 inches.

139. A certain number, consisting of two digits, is equal to twice the product of its digits; and if 27 be added to the number, its digits will be inverted. Required the number.

Ans. 36.

140. The sum of three numbers in arithmetical progression is $\frac{3}{2}$, and the sum of their reciprocals is $7\frac{1}{3}$; what are the numbers? *Ans.* $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$.

141. Find three geometrical means between $\frac{1}{2}$ and $\frac{2}{9}$.

Ans. $\frac{1}{6}\sqrt{6}$, $\frac{1}{3}$, and $\frac{1}{9}\sqrt{6}$.

142. There are three quantities related as follows: the sum of the squares of the first and second, added to the first and second, is 18; the sum of the squares of the first and third, added to the first and third, is 26; and the sum of the squares

of the second and third, added to the second and third, is 32 Required the quantities.

$$Ans. \begin{cases} \text{1st, } 2, \text{ or } -3; \\ \text{2d, } 3, \text{ or } -4; \\ \text{3d, } 4, \text{ or } -5. \end{cases}$$

143. The compound interest of a certain sum of money for 3 years was \$364 ; and the interest for the first year was to the interest which accrued the third year as 25 to 36. Required the sum at interest. *Ans.* \$500.

144. Find 3 numbers in arithmetical progression such that their sum shall be 36, and 4 added to the product of the extremes shall be equal to the square of the mean.

Ans. 10, 12, and 14.

145. Find three numbers in geometrical progression whose sum shall be 52, and the sum of the extremes to the square of the mean as 10 to 36. *Ans.* 4, 12, and 36.

146. There are three numbers in geometrical progression; their continued product is 1, and the difference of the first and second is to the difference of the second and third as 1 to 3; what are the numbers? *Ans.* $\frac{1}{3}$, 1, and 3.

147. There are three numbers in geometrical progression, such that three times the first, twice the second, and once the third, taken in order, form an arithmetical series; and also the first, the second increased by 8, and the third, taken in order, form an arithmetical series. What are the numbers?

Ans. 4, 12, and 36.